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Regularity for Solutions of Nonlocal Fully Nonlinear Parabolic Equations and Free Boundaries on two Dimensional Cones

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Regularity for Solutions of Nonlocal Fully Nonlinear Parabolic Equations and Free Boundaries on two Dimensional Cones

by

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Dedicated to Carlos, Rhaitza and Marino.

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On the first part, we consider nonlinear operators I depending on a family of nonlocal linear operators,

$$\begin{aligned} Iu(x,t) &:= I\left(x,t,(L_{K,b}u(x,t))_{K,b}\right), \\ L_{K,b}u(x,t) &:= \int_{\mathbb{R}^n} \left(u(x+y,t)-u(x,t)-Du(x,t)\cdot y\chi_{B_1}(y)\right)K(y)dy \\ &\quad + bDu(x,t) \end{aligned}$$

We study the solutions of the Dirichlet initial and boundary value problems,

$$\begin{aligned} u_t - Iu &= f \text{ in } \Omega\times(-1,0], \\ u &= g \text{ in } \Omega\times\{-1\}\cup(\mathbb{R}^n\setminus\Omega)\times[-1,0]. \end{aligned}$$

We do not assume even symmetry for the kernels. The odd part bring some sort of nonlocal drift term, which in principle competes against the regularization of the solution.

Existence and uniqueness is established for viscosity solutions. Several Hölder estimates are established for u and its derivatives under special assumptions. Moreover, the estimates remain uniform as the order of the equation approaches the second order case. This allows to consider our results as an extension of the classical theory of second order fully nonlinear equations.

On the second part, we study two phase problems posed over a two dimensional cone generated by a smooth curve γ on the unit sphere. We show that when length$(\gamma) < 2\pi$ the free boundary avoids the vertex of the cone. When length$(\gamma) \geq 2\pi$ we provide examples of minimizers such that the vertex belongs to the free boundary.

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Part I

Nonlocal Parabolic Equations

Chapter 1

Introduction

1.1 Motivation from Optimal Stochastic Control

Our goal in this part is to answer the following question. Does the following (backwards) problem has a (classical) solution?

$$\begin{aligned}
-u_t &= \inf_{\beta\in B}\sup_{\alpha\in A} L_{K_{\alpha,\beta}}u \text{ in } B_1\times\mathbb{R}^-,\\
u &= g \qquad\qquad\quad\ \text{in } (B_1^c\times\mathbb{R}^-)\cup(\mathbb{R}^n\times\{0\}),
\end{aligned}$$

where,

$$\begin{aligned}
L_{K_{\alpha,\beta}}u(x,t) &:= \int (u(x+y,t)-u(x,t)-Du(x,t)\cdot y\chi_{B_1}(y))K_{\alpha,\beta}(y)dy,\\
K_{\alpha,\beta}(y) &\geq 0 \text{ for every } \alpha,\beta\in A\times B.
\end{aligned}$$

This model arises, for instance, in optimal stochastic control in the way we are about to describe.

Consider a game taking place between players A and B. The game starts at time $t = t_0 < 0$ with a token placed at some $x_0 \in B_1$. At each time A and B play by the following rules to move the token from $x(t)$ to $x(t+dt)$.

1. B fixes $\beta \in B$ and let A know about his choice,

2. A fixes $\alpha \in A$,

3. The token get displaced by the Lévy processes $X_{\alpha,\beta}(t)dt$ with pure jumps determined by the Lévy measures $K_{\alpha,\beta}$ which are known for both players.

The game ends when the token exits B_1 or $t \geq 0$. Whenever this happens, B has to pay to A a predetermined price $U = g(\text{exit position of the token})$ which is known by both players since the beginning of the game.

Now you may wonder if:

1. Are there optimal strategies for players A and B?

2. What is the expected fair price $u(x_0, t_0) = \mathbb{E}[U(x_0, t_0)]$, player A should pay to player B, for a game starting at $(x_0, t_0) \in B_1 \times \mathbb{R}^-$?

Actually these two questions are very related. The optimal strategy can be obtained from the knowledge of u by a dynamic programing principle.

We will not go into the details of defining what is a Lévy process. A good reference is the book by W. M. Fleming and H. M. Soner [26]. What we will use about it is that, by fixing $(\alpha, \beta) \in A \times B$, $u(x,t)$ gets modified in the following way, also known as Ito's formula,

$$u(x, t + dt) = \mathbb{E}\left[u(x + X_{\alpha,\beta}dt, t + dt)\right] \sim u(x,t) + L_{K_{\alpha,\beta}}u(x,t)dt. \tag{1.1.1}$$

Important observations can be made about L_K by decomposing $K = K_e + K_o$ in its even and odd part respectively. $K \geq 0$ implies $K_e \geq 0$ so that $L_{K_e}u$

measures a deviation of u from an average of itself. This is a diffusive term. Formally,

$$L_{K_e}u(x,t) = \int (u(x+y,t) - u(x,t))K_e(y)dy.$$

To say that $-u_t = L_{K_e}u$ means that u will try to accommodate to some average of itself as time goes backwards.

On the other hand, K_o change signs in opposite directions so that $L_{K_o}u$ measures an average of how u grows in each direction. This is a drift term. Also formally,

$$L_{K_o}u(x,t) = \int \left(\frac{u(x+y,t) - u(x-y,t)}{2|y|} \frac{y}{|y|} - Du(x,t)\chi_{B_1}(y) \right) \cdot yK_o(y)dy.$$

To say that $-u_t = L_{K_o}u$ means that the values of u follow some sort of (constant) nonlocal flow dictated by the vector valued kernel $yK_o(y)$.

Going back to the optimal strategy, assume that the token is at the position $x(t)$ at time t, B has already fixed $\beta \in B$ and A knows $u(\cdot, t+dt)$. Then, the best move for player A is pretty clear. It has is to choose $\alpha \in A$, using its knowledge of kernel $K_{\alpha,\beta}$, such that it maximizes the expected value of its earnings in the next step by using (1.1.1). See figure 1.1.

By applying this strategy, A will expect the following earnings for a game starting at $(x(t), t)$,

$$u_\beta(x(t),t) = \sup_{\alpha \in A} \mathbb{E}[u(x(t) + X_{\alpha,\beta}dt, t+dt)].$$

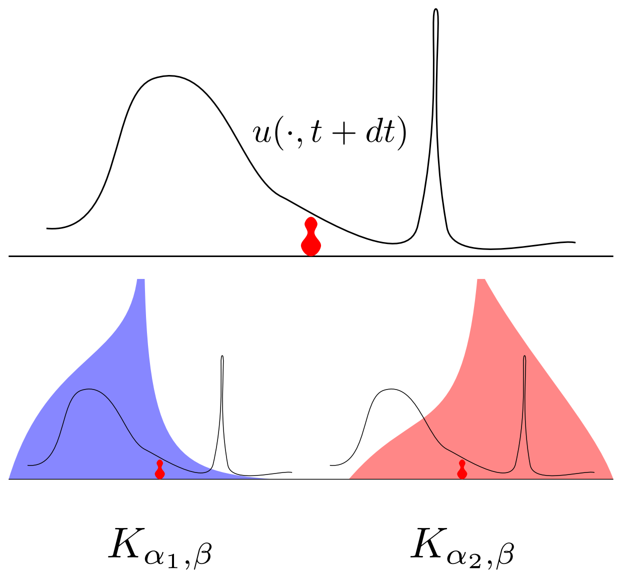


Figure 1.1: Two possible options for player A.

But B knows that A will follow this strategy, so he will choose $\beta \in B$ in order to minimize the previous quantity. This gives the following dynamic programing principle for u,

$$\begin{aligned} u(x(t), t) &= \inf_{\beta \in B} \sup_{\alpha \in A} \mathbb{E}[u(x(t) + X_{\alpha,\beta} dt, t + dt)], \\ \frac{u(x,t) - u(x, t+dt)}{dt} &= \inf_{\beta \in B} \sup_{\alpha \in A} \mathbb{E}\left[\frac{u(x + X_{\alpha,\beta} dt, t+dt) - u(x, t+dt)}{dt}\right], \\ -u_t &\sim \inf_{\beta \in B} \sup_{\alpha \in A} L_{K_{\alpha,\beta}} u. \end{aligned}$$

This is finally how our equation comes into play.

As we can see an optimal strategy for both players can be contructed by solving the integro-differential equation at hand. As good mathematicians they are, A and B know that the consistency of any numerical scheme they use to solve the equation relies on the regularity that the solution may have. That is a good reason to study regularity theory.

Once motivated our equation we will consider from now on the problem foward in time. That is,

$$\begin{aligned} u_t &= \inf_{\beta \in B} \sup_{\alpha \in A} L_{K_{\alpha,\beta}} u \text{ in } B_1 \times (a,b], \\ u &= g \qquad\qquad\qquad \text{in } (B_1^c \times (a,b]) \cup (\mathbb{R}^n \times \{a\}). \end{aligned}$$

1.2 Nonlocal vs. Local

Integrodifferential elliptic operators certainly resemble second order elliptic operators as the Laplacian or the $\inf_{\beta \in B} \sup_{\alpha \in A} \operatorname{tr}(A_{\alpha,\beta} D^2 u)$ where each $A_{\alpha,\beta}$ is a symmetric positive definitive matrix. Nonlocal operators have also been studied since long time ago in probabiliy and potential theory. However, it was not until recent years that new results have shown that the second order theory of fully nonlinear equations can be extended to the nonlocal counterpart. To mention some of the references, see [3–5, 10, 12, 13, 27] for the analytical point of view and [6–8] for the probabilistic point of view.

The nonlocality can be sometimes beneficial or challenging for the proofs. Some of the proofs we know leading to Hölder regularity for fully nonlinear parabolic equations rely on a nonlocal principle, namely the Mean Value formula or a Point Estimate or L^ε Lemma, see [17, 29, 33, 37, 38] and the discussion on the next section. On the good side, the definition of L_K has already built in a Mean Value Formula. Recall that for K even, $u_t - L_K u = 0$ means that u will try to accommodate to some sort of average of itself as times goes on. This is one important tool which allowed L. Silvestre to give a very elegant proof of Hölder regularity for Hamilton-Jacobi equations with critical fractional diffusion, see [35].

The work in [35], initially done for operators of order one, can also be extended to fully nonlinear operators of any order $\sigma \in (0,2)$. However, there was one price to pay with this proof, the estimate does not remain uniform as the order σ goes to two. To recover the second order theory in the limit it was necessary to follow more closely the classical proofs. This was actually started for elliptic equations in a serie of papers by L. Caffarelli and L. Silvestre, see [10–12]. Our contribution, together with G. Dávila, was to develop the time dependent counterpart in [31].

To finish this section let us illustrate with an example one of the main challenges that we encounter for integrodifferential, time dependent equations. Consider u to be the solution of the fractional heat equation,

$$\begin{aligned} u_t + (-\Delta)^\sigma u = 0 \text{ in } B_1 \times (-1,1], \\ u = g \text{ in } B_1^c \times (-1,1] \cup \mathbb{R}^n \times \{-1\}. \end{aligned}$$

Initially we fix g to be zero in $\mathbb{R}^n \times [-1,0]$ such that the equation on such interval of time gets trivially solved by $u = 0$. After time zero we may add to g the characteristic function of a set away from B_1. This contribution is immediately felt inside the domain of the equation and in fact g becomes a sub solution of the fractional heat equation. Adding a positive and sufficiently

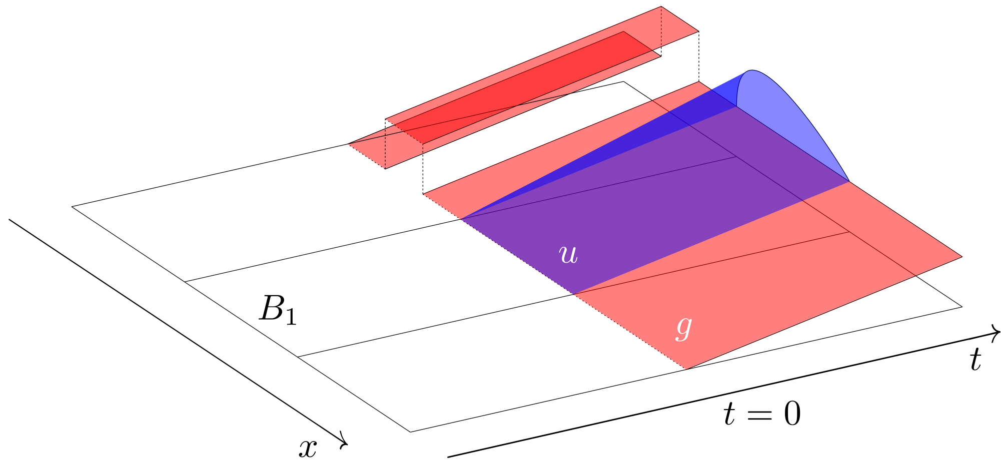


Figure 1.2: Counterexample for C^1 regularity in time.

small multiple of the function $t \mapsto t^+$ to g will not change it from being a sub solution. See figure 1.2. By the comparison principle we know that u at time zero can not be C^1 in time.

We will see that to avoid this phenomena, eventually we will have to impose some control on the time behavior of the boundary data.

1.3 Strategy to Prove Hölder Regularity

Now that we have motivated the type of equations we will be studying, lets go back to real analysis and sketch the main strategy we will use to prove regularity for the solutions. Consider the well known (forward) heat equation with constant drift,

$$u_t + b \cdot Du - \Delta u = 0 \text{ in } B_1 \times (-1, 0].$$

An α-Hölder modulus of continuity at the origin, namely,

$$|u(x,t)-u(0,0)| \leq C(|x|+|t|^{1/2})^\alpha,$$

is equivalent to a geometric decay of the oscillation. That is to say that for all $k\in\mathbb{N}$,

$$\operatorname*{osc}_{B_{\kappa^k}\times[-\kappa^{2k},0]} u \leq (1-\theta)^k \operatorname*{osc}_{B_1\times(-1,0]} u,$$

for some $\kappa,\theta\in(0,1)$. Then α can be made equal to $\left|\frac{\ln(1-\theta)}{\ln\kappa}\right|$.

By changing variables we can assume that the drift term b is zero. Abusing the notation, $u(x,t)=u(x+bt,t)$ solves the heat equation without drift.

By the scaling of the equation it is enough to prove the decay at scale one provided that $\operatorname{osc}_{B_1\times(-1,0]} u \leq 1$,

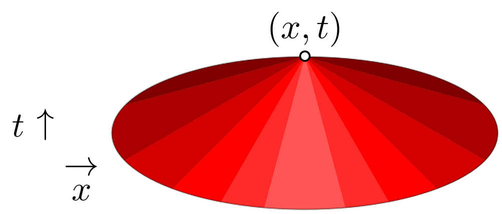
$$\operatorname*{osc}_{B_\kappa\times[-\kappa^2,0]} u \leq (1-\theta).$$

Now we have two possibilities. Either u lies above or below the 1/2-level set at least half of the measure in some cylinder $B_r\times(a,b]$ which will be fixed in a moment. Lets assume without loss of generality that the fist case holds. The Mean Value Formula, see figure 1.3, then will tells us that,

$$\frac{|\{u\geq 1/2\}\cap B_r\times(a,b]|}{|B_r\times(a,b]|} \geq \frac{1}{2} \Rightarrow \inf_{B_\kappa\times[-\kappa^2,0]} u \geq \theta.$$

In order to have this implication we choose a radius $2R$ such that the heat ball $E(0,0,2R)$ is contained in the domain of the equation $B_1\times(-1,0]$. Then

$$u(x,t) = \frac{1}{(4r)^n} \iint_{E(x,t,r)} u(y,s) \frac{|y-x|^2}{|s-t|^2} dyds$$



$$E(x,t,r) = \left\{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \leq t, \exp\left(-\frac{|y-x|^2}{4|s-t|^2}\right) \geq (4\pi|s-t|)^{n/2}\right\}$$

Figure 1.3: Mean value formlua.

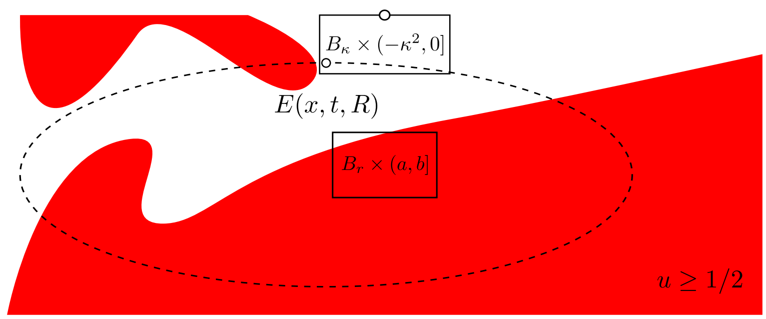


Figure 1.4: How to fix κ and the cylinder $B_r \times (a,b]$.

we choose κ sufficiently small such that for every $(x,t) \in B_\kappa \times (-\kappa^2, 0]$ each one of the heat balls $E(y,s,R)$ are also contained in in the domain of the equation. Finally, we fix the cylinder $B_r \times (a,b]$ such that it is contained in the intersection of all the previous heat balls and it is strictly disjoint with $B_\kappa \times (-\kappa^2, 0]$. In the case that the previous intersection has empty interior we just need to choose κ even smaller to solve the difficulty. See figure 1.4.

Finally, for every $(x,t) \in B_\kappa \times (-\kappa^2, 0]$ we have that,

$$\begin{aligned}
u(x,t) &= \frac{1}{(4R)^n} \iint_{E(x,t,R)} u(y,s) \frac{|y-x|^2}{|s-t|^2} dyds,\\
&\geq \frac{1}{2(4R)^n} \inf_{\substack{(x,t)\in B_\kappa \times(-\kappa^2,0]\\ (y,s)\in B_r\times(a,b]}} \frac{|y-x|^2}{|s-t|^2} |\{u \geq 1/2\} \cap B_r \times (a,b]|,\\
&=: \theta > 0.
\end{aligned}$$

This completes the diminish of oscillation which implies the Hölder regularity of the solution.

This is not necessarily the shortest proof we know for the regularity of the heat equation. However is the one we can adapt to prove Hölder regularity for fully nonlinear equations as the one we saw in the previous section. The new challenges are that:

1. There might not be a single change of variables that gets rid of the drift term.

2. for the fully nonlinear operator $Iu = \inf_{\beta\in B} \sup_{\alpha \in A} L_{K_{\alpha,\beta}}$ we do not count with a Mean Value formula as for the heat equation. This role is instead fulfilled by a Point Estimate Theorem which we prove in Section 5.1.

3. $u_t - L_K u = f$ is not scale invariant if K is not even, we will discuss this in Section 2.1.1.

4. Assuming that the solution a global bound is not a sufficiently strong hypothesis to iterate in a diminish of oscillation Lemma. As we scale

and try to reapply such Lemma we found that the tails of the original solution, which can not be ignored in the nonlocal setting, start growing with a polynomial rate. Therefore, the hypothesis of our diminish of oscillation lemmas has to be strengthened by allowing this grow in the initial hypothesis. As $\sigma \in [1,2)$ we will usually assume that, after some renormalization,

$$|u| \leq 1/2 \quad \text{in } B_1 \times (-1,0],$$
$$|u(y,s)| \leq |y|^{1/2} \text{ for } y \in \mathbb{R}^n \setminus B_1.$$

Then a diminish of oscillation Lemma which implies that $\operatorname{osc}_{B_\kappa \times (-\kappa^2,0]} u \leq (1-\theta)$ would need that $(1-\theta) - \kappa^{1/2} \geq \theta/2 > 0$ in order to be iterated. But this is always reasonable as κ and θ can be made smaller if necessary.

1.4 Outline

On the Preliminary Chapter 2 we establish the fully nonlinear, integrodifferential operators we will study. This is done by considering functions which depend on all possible linear operators. By scaling considerations, we will see that some further restrictions on the kernel allow the operator to be uniformly ellipticity at smaller scales. Following the examination of some examples we proceed to define the notion of viscosity solutions and mention some direct properties about them.

On Chapter 3 we discuss the qualitative behavior of the viscosity solutions as the Stability Theorem 3.1.1, the Maximum Principle Theorem 3.2.1, the uniform ellipticity identity for viscosity solutions Theorem 3.2.3, The Comparison Principle Theorem 3.2.8 and the existence and uniqueness of viscosity solutions of the Dirchlet problem by Perron's Method, Theorem 3.3.4.

Chapter 4 treats estimates like the classical Alexandrov-Bakelman-Pucci-Krylov-Tso. They are the first step towards a Point Estimate for equations of order $\sigma \in [1, 2)$ with constants that remain uniform in σ. As we already saw this in this introduction, this is a key step towards the Hölder regularity estimates. The challenge relies in the fact that the literal Alexandrov-Bakelman-Pucci-Krylov-Tso estimate does not hold in the nonlocal setting as the Monge-Ampère measure of the convex envelope becomes singular when the equation is of order smaller than two. The new idea introduced in [10] is to cover the contact set with pieces where we can control the detachment of the solution and its convex envelope. In order to do this we rely on a version of a weak Point Estimate which we learned from [35].

Chapter 5 proves the Point Estimate result we need to prove Hölder regularity, which is also done in that chapter. The proof uses a Calderon-Zygmund type of decomposition of the level sets of u in order to show that their measure decrease in a universal way. The decomposition is not the standard one used for parabolic equations by subdividing cubes in half in space and time intervals by four. It needs instead to keep track of the scaling of the equation which not of order two.

As a consequence of the Hölder regulatiry result we also get further regularity for the derivatives of solutions of translation invariant equations. One final result which we treat in this chapter is the Oscillation Lemma 5.5.1. It gives a control of the supremum of a sub solution by an integral quantity of itself. This allows us to improve the Comparison Principle treated in Chapter 3.

Chapter 6 handles regularity results by approximation methods as the Cordes-Nirenberg estimates. As these type of result rely on compactness principles we need to go a little bit further into functional analysis implications of our definitions. One interesting result that we deduce in this chapter is that solutions of translation invariant equations can be approximated by regular solutions of close by equations. This is one example where nonlocal problems simplify the analysis.

The final Chapter of this part studies concave equations, which means that the operator is the infimum of a family of linear operators. We present an estimate which controls the size of every linear operator applied to the solution. This says in some sense that viscosity solutions are also classical. So far we do not know if an Evans-Krylov type of estimate holds for these equations.

1.5 Notation

1. Our domains will be always contained in $\mathbb{R}^n \times \mathbb{R}$ and usually denoted by $\Omega \times (a, b]$ for some $\Omega \subseteq \mathbb{R}^n$ open and bounded. The first variable in $\mathbb{R}^n$ is called the **spatial variable** and the second variable in $\mathbb{R}$ is called the

time variable.

2. $\Omega^c = \mathbb{R}^n \setminus \Omega$.

3. Given a set A we denote the **characteristic function** of A by $\chi_A(y) = 1$, for $y \in A$ and zero otherwise.

4. The canonical base of $\mathbb{R}^n$ is denoted by $e_1, \ldots, e_n$ and the canonical base of $\mathbb{R}^n \times \mathbb{R}$ is denoted by $e_1, \ldots, e_n, e_t$. The **norm** and **inner product** in $\mathbb{R}^n$ are denoted by $|x| := \sqrt{x_1^2 + \ldots + x_n^2}$ and $x \cdot y = x_1 y_1 + \ldots + x_n y_n$.

5. The **ball** of radius r and center x in $\mathbb{R}^n$ is denoted by $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$. Usually, when the center is omitted it means that the ball is centered at the origin.

6. The **cube** of side length r and center x in $\mathbb{R}^n$ is denoted by $Q_r(x) := (x_1 - r/2, x_1 + r/2) \times \ldots \times (x_n - r/2, x_n + r/2)$. Usually, when the center is omitted it means that the cube is centered at the origin.

7. The **cylinder** of radius r, height τ and center (x, t) in $\mathbb{R}^n \times \mathbb{R}$ is denoted by $C_{r,\tau}(x, t) := B_r(x) \times (t - \tau, t]$. Usually when the center is omitted it means that the cylinder is centered at the origin.

8. The **box** of side length r, height τ and center (x, t) in $\mathbb{R}^n \times \mathbb{R}$ is denoted by $K_{r,\tau}(x, t) := Q_r(x) \times (t - \tau, t]$. Usually when the center is omitted it means that the box is centered at the origin.

9. Whenever we have a set $A \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we denote the **scaled set** αA by $\alpha A := \{y \in \mathbb{R}^n : \alpha y \in A\}$. In particular a set A is even if $A = -A$.

10. The **parabolic topology** on $\mathbb{R}^n \times \mathbb{R}$ consists of the one generated by neighborhoods of the form $C_{r,\tau}(x,t)$ with respect to the point (x,t). We use $(x_i, t_i) \to (x, t^-)$ to denote a sequence converging to the point (x,t) with respect to this topology.

11. The **parabolic nonlocal boundary**, suitable for our Dirichlet problem on a domain $\Omega \times (a,b]$, is

$$\partial_p(\Omega \times (a,b]) := (\Omega^c \times (a,b]) \cup (\mathbb{R}^n \times \{a\}).$$

12. The spaces of **upper and lower semicontinuous** functions are denoted by USC and LSC respectively. We will always assume that they are locally bounded from above, in the case of USC, and from below in the case of LSC.

13. **Hölder spaces** are denoted by $C^{j,\alpha}$ for $j \in \mathbb{N}$ and $\alpha \in [0,1]$. The Hölder spaces $C^{j,\alpha}_x$ and $C^{k,\beta}_t$ are used for functions depending in space and time which are $C^{j,\alpha}$ in space and $C^{k,\beta}$ in time.

14. For an integer k, a function $u : B_r(x) \to \mathbb{R}$ is **punctually k^{th} order differentiable at** x if there exists a k^{th} order polynomial P such that $u(y) = P(y-x) + o(|y-x|^k)$. In that case we denote it also by $u \in C^k(x)$ and P is the unique k^{th} order Taylor expansion of u at x.

15. For a function depending on a space variable $x \in \mathbb{R}^n$ and a time variable t we distinguish Du as the gradient with respect to x and u_t as the time derivative of u. Given that we work with respect to the parabolic topology we will also use,

$$u_{t^-}(x_0, t_0) := \lim_{\tau \searrow 0} \frac{u(x_0, t_0) - u(x_0, t_0 - \tau)}{\tau}.$$

16. For $\sigma \in (0, 2)$, the **weighted space** $L^1(\omega_\sigma)$ with respect to

$$\omega_\sigma(y) := \min(1, |y|^{-(n+\sigma)})$$

consists of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|u\|_{L^1(\omega_\sigma)} := \int_{\mathbb{R}} |u(y)| \omega_\sigma(y) dy < \infty.$$

Chapter 2

Preliminaries

We give precise definitions of the fully nonlinear integrodifferential operators that we will consider. For second order equations, all the information required to define a second order operator is given by the hessian. In the nonlocal setting we consider instead all the possible linear operators of a given order $\sigma \in (0, 2)$ and define a nonlinear operator as any function depending on this infinite dimensional vector. This appeared is the case treated for instance in [4].

Uniform ellipticity gets defined by controlling the monotonicity and boundedness of the operator in a uniform way. Scaling a linear operator will bring drift terms which were not explicitly present before. This consideration made us impose that the order σ has to be at least one in order to compete with the drift term which propagates discontinuities. Also, in order to prove results that can be iterated at smaller scales we need to consider since the beginning linear operators with a drift term.

On the second half of this Chapter we define viscosity (sub and super) solutions. This definition is given in terms of the comparison principle with regular solutions. Finally we point out some properties which follow from this definition.

2.1 Nonlocal Uniformly Elliptic Operators

Given a measurable kernel $K : \mathbb{R}^n \to [0,\infty)$ and a vector $b \in \mathbb{R}^n$, the nonlocal linear operator $L_{K,b}$ is defined by

$$\begin{aligned} L_{K,b}u(x,t) &:= \int \delta u(x,t;y)K(y)dy + b \cdot Du(x,t), \qquad (2.1.1)\\ \delta u(x,t;y) &:= u(x+y,t) - u(x,t) - Du(x,t)\cdot y\chi_{B_1}(y). \end{aligned}$$

We will also denote $L_{K,0}$ and $L_{0,b}$ by L_K and $b \cdot D$ respectively.

A general assumption we will use for K, which makes the previous integral convergent whenever u is sufficiently smooth and integrable, is the following,

$$\int \min(|y|^2, 1)K(y)dy < \infty. \qquad (2.1.2)$$

We call $\mathbb{K}^+$ to the family of kernels satisfying the previous restriction. A more specific hypothesis that implies the previous one in a uniform way and that we will constantly use is that,

$$\frac{(2-\sigma)\lambda\chi_{B_\rho}(y)}{|y|^{n+\sigma}} \leq K(y) \leq \frac{(2-\sigma)\Lambda}{|y|^{n+\sigma}} \qquad (2.1.3)$$

for some parameters $\sigma \in (0,2)$, $\rho > 0$ and $\Lambda \geq \lambda > 0$. We denote by $\mathcal{K}_0 = \mathcal{K}_0(\sigma,\rho,\lambda,\Lambda)$ the family of all measurable kernels satisfying (2.1.3) and we say that they are **uniformly elliptic** with respect to the given parameters.

Given $\mathcal{L} \subseteq \mathbb{K}^+ \times \mathbb{R}^n$, a function $I : \Omega \times (a,b] \times \mathbb{R}^{\mathcal{L}} \to \mathbb{R}$ determines a nonlocal operator of order σ by,

$$Iu(y,s) := I\left(y, s, (L_{K,b}u(y,s))_{(K,b)\in\mathcal{L}}\right).$$

We will write $I(y,s,l_{K,b})$ and $(l_{K,b})$ instead of $I(y,s,(l_{K,b})_{(K,b)\in\mathcal{L}})$ and $(l_{K,b})_{(K,b)\in\mathcal{L}}$ if there is no chance of confusion. Furthermore, we will denote $l_{K,0}$ and $l_{0,b}$ by l_K and p_b respectively.

I is **(degenerate) elliptic** if it is monotone (nondecreasing) increasing in $(l_{K,b})$. I is said to be **translation invariant in space or time** if the function I does not depend on the variable y or s, respectively. Translation invariant, without making reference the space or time variable, means that it is translation invariant with respect to both. Finally, I is said to be **(semi)continuous** if the function I is (semi)continuous when $\Omega \times (a,b] \times \mathbb{R}^{\mathcal{L}}$ is equipped with the L^∞ norm.

Before giving the definition of **uniform ellipticity** let us analyze the behavior that scaling has over these linear operators. As we will see it will impose some further restrictions which will be frequently used.

2.1.1 Scaling

An important ingredient of the regularity theory is scale invariance. A diminish of oscillation estimate works to prove regularity of the solution because they also hold also at smaller scales.

Given $\sigma \in (0,2)$, and u satisfying the non-homogeneous linear equation,

$$u_t - L_K u = f \text{ in } \Omega \times (a,b],$$

we consider a rescaling of the form $\tilde{u}(y,s) := r^{-\sigma} u(ry, r^\sigma s)$ with $r \in (0,1]$. By the change of variable formula it satisfies in $r^{-1}\Omega \times (r^{-\sigma}a, r^{-\sigma}b]$,

$$\tilde{u}_t - L_{r^{n+\sigma}K(r\cdot)}\tilde{u} - \left(r^{\sigma-1}\int_{B_1\setminus B_r} yK(y)dy\right)\cdot D\tilde{u} = f(r\cdot). \tag{2.1.4}$$

It comes immediately to our attention the gradient term which was not explicitly present in L_K and which depends on the odd part of K. This explains why we included the gradient variable in I. Also the diffusion, contained in $L_{r^{n+\sigma}K(r\cdot)}$, competes against the drift term if $\sigma \in [1,2)$.

So, we consider now the scaling of $L_{K,b}$. In order to have a bounded drift at small scales we need to assume that

$$\sup_{r\in(0,1]} r^{\sigma-1}\left|b + \int_{B_1\setminus B_r} yK(y)dy\right| < \infty.$$

Definition 2.1.1. ***For*** $\sigma \in [1,2)$, *let* $\mathcal{L}_0(\sigma,\rho,\lambda,\Lambda,\beta)$ *be the family of pairs* $(K,b) \in \mathcal{K}_0(\sigma,\rho,\lambda,\Lambda)\times\mathbb{R}^n$ *such that,*

$$\sup_{r\in(0,1]} r^{\sigma-1}\left|b + \int_{B_1\setminus B_r} yK(y)dy\right| \leq \beta.$$

The important thing about $\mathcal{L}_0$ is that at scales smaller than one it remains inside itself. Indeed, let $L_{K,b}$ with $(K,b)\in\mathcal{L}_0$, then proceeding as before we need to verify that for $r\in(0,1]$, $L_{K^r,b^r}\in\mathcal{L}_0$ where,

$$\begin{aligned} K^r(y) &:= r^{n+\sigma}K(ry),\\ b^r &:= r^{\sigma-1}\left(b + \int_{B_1\setminus B_r} yK(y)dy\right). \end{aligned}$$

Clearly $K^r \in \mathcal{K}_0(\sigma, \rho/r, \lambda, \Lambda) \subseteq \mathcal{K}_0(\sigma, \rho, \lambda, \Lambda)$ meanwhile,

$$\sup_{\rho\in(0,1]} \rho^{\sigma-1} \left| b^r + \int_{B_1\setminus B_\rho} yK^r(y)dy \right| =$$
$$\sup_{\rho\in(0,1]} \rho^{\sigma-1} \left| r^{\sigma-1} \left(b + \int_{B_1\setminus B_r} yK(y)dy \right) + \int_{B_1\setminus B_\rho} yr^{n+\sigma}K(ry)dy \right| =$$
$$\sup_{\rho\in(0,1]} (r\rho)^{\sigma-1} \left| b + \int_{B_1\setminus B_{r\rho}} yK(y)dy \right| \leq \beta.$$

On the other hand, this is not necessarily true for scalings larger than one. The reason being that $\mathcal{K}_0(\sigma, \rho, \lambda, \Lambda) \not\subseteq \mathcal{K}_0(\sigma, \rho/r, \lambda, \Lambda)$. However, it is true that $L_{K^r,b^r} \in \mathcal{L}_0(\sigma, \rho, \lambda, \Lambda)$ if and only if $L_{K,b} \in \mathcal{L}_0(\sigma, \rho/r, \lambda, \Lambda)$ for any $r > 0$.

For $\sigma > 1$, it is enough to assume that $|b| \leq \beta$ in order to have a bounded drift. Indeed, for $r \in (0, 1]$,

$$\begin{aligned} r^{\sigma-1} \left| b + \int_{B_1\setminus B_r} yK(y)dy \right| &\leq r^{\sigma-1}\beta + \frac{|\partial B_1|(2-\sigma)\Lambda}{(\sigma-1)}(1 - r^{\sigma-1}), \\ &\leq \beta + \frac{|\partial B_1|(2-\sigma)\Lambda}{(\sigma-1)}. \end{aligned}$$

The problems with this assumption is that, on one side, it is not scale invariant and on the other it make the constants to blow up as $\sigma \searrow 1$.

As an example, consider $\mathcal{K}_e = \mathcal{K}_e(\sigma, \rho, \lambda, \Lambda) \subseteq \mathcal{K}_0(\sigma, \rho, \lambda, \Lambda)$ being defined as the family of all the kernels $K \in \mathcal{K}_0$ which are even, namely $K(y) = K(-y)$. Given $L_{K,b} \in \mathcal{K}_e \times B_\beta \subseteq \mathcal{L}$, the scaling of the drift term $b \cdot D$ and the nonlocal diffusion L_K get decoupled but, as before, the diffusion competes against the drift at smaller scales only if $\sigma \geq 1$. These assumptions allow us to handle for instance the following critical equation, previously studied in [35],

$$u_t - \Delta^{1/2}u - |Du| = 0. \tag{2.1.5}$$

What it is remarkable about this equation, is that one can clearly appreciate that the drift and the diffusion compete with the same order.

Definition 2.1.2 (Uniformly Ellipticity). *For $\mathcal{L} \subseteq \mathcal{L}_0(\sigma, \rho, \lambda, \Lambda, \beta)$ and $I : \Omega \times (a, b] \times \mathbb{R}^{\mathcal{L}} \to \mathbb{R}$, we say that I is uniformly elliptic with respect to $\mathcal{L}$ if for every $\left(y, s, l^{(1)}_{K,b}\right), \left(y, s, l^{(2)}_{K,b}\right) \in \mathbb{R}^{\mathcal{L}}$,*

$$\inf_{(K,b)\in\mathcal{L}} \left(l^{(1)}_{K,b} - l^{(2)}_{K,b}\right) \leq I\left(y, s, l^{(1)}_{K,b}\right) - I\left(y, s, l^{(2)}_{K,b}\right) \leq \sup_{(K,b)\in\mathcal{L}} \left(l^{(1)}_{K,b} - l^{(2)}_{K,b}\right). \tag{2.1.6}$$

Whenever we omit the family $\mathcal{L}$ we are just using that I is uniformly elliptic with respect to $\mathcal{L}_0$.

The uniform ellipticity identity implies that I is Lipschitz in $\mathbb{R}^{\mathcal{L}}$, uniformly in $\Omega \times (a, b]$, namely,

$$\sup_{(y,s)\in\Omega\times(a,b]} \left| I\left(y, s, l^{(1)}_{K,b}\right) - I\left(y, s, l^{(2)}_{K,b}\right) \right| \leq \left\| \left(l^{(1)}_{K,b} - l^{(2)}_{K,b}\right) \right\|_{\infty}.$$

We finish this section showing how to rescale a more general operator $I : \Omega \times (a, b] \times \mathbb{R}^{\mathcal{L}} \to \mathbb{R}$. For $r > 0$, let,

$$\begin{aligned}
I^r(y, s, l_{K,b}) &:= I(ry, r^\sigma s, (l_{K^r,b^r})_{(K,b)\in\mathbb{R}^{\mathcal{L}}}),\\
K^r(y) &:= r^{n+\sigma}K(ry),\\
b^r &:= \begin{cases} r^{\sigma-1}\left(b + \int_{B_1\setminus B_r} yK(y)dy\right) & \text{if } r \leq 1,\\ r^{\sigma-1}\left(b - \int_{B_r\setminus B_1} yK(y)dy\right) & \text{if } r > 1,\end{cases}\\
\tilde{f}(y, s) &:= f(ry, r^\sigma s).
\end{aligned}$$

If u satisfies,

$$u_t - Iu = f \text{ in } \Omega \times (a, b]$$

then $\tilde{u}(y, s) := r^{-\sigma} u(ry, r^\sigma s)$ satisfies

$$\tilde{u}_t - I^r \tilde{u} = \tilde{f} \text{ in } r^{-1}\Omega \times (r^{-\sigma} a, r^\sigma b].$$

2.1.2 Examples

For $\sigma \in (0, 2)$, a **fractional power of the laplacian** $\Delta^{\sigma/2} = -(-\Delta)^{\sigma/2}$ is defined as the linear operator with homogeneous kernel $K_{\Delta^{\sigma/2}}(y) := C_{n,\sigma}|y|^{n+\sigma}$. The constant $C_{n,\sigma}$ is used to have the following identity on the Fourier side, $\widehat{(-\Delta)^\sigma} = |\xi|^\sigma$. The only important information we will use about the constant $C_{n,\sigma}$ in this work is that $C_{n,\sigma}/(2-\sigma)$ remains uniformly bounded for $\sigma \in [1, 2)$. This explains why we have included the constant $(2 - \sigma)$ in (2.1.3), so that we approach the second order theory as $\sigma \to 2$.

Linear operators with variable coefficients are those defined as in (2.1.1) replacing $K(y)$ and b by $K(x, t; y)$ and $b(x, t)$ respectively. They are also an example of the type of operators that can be obtained by a function I which depends on the position in $\Omega \times (a, b]$.

Whenever I is split as $I(y, s, l_{K,b}) = V(y, s, l_k) + H(y, s, p_b)$ it means that we are dealing with operators where H can be considered as a **Hamiltonian depending on the gradient** and V contains the **viscosity term**. For example, $I(y, s, l_{K,b}) = l_{K_{\Delta^{1/2}}} + \sup_{b \in B_1} |p_b|$ refers to the critical equation (2.1.5).

Operators obtained by **inf and sup combinations** of linear operators are relevant for stochastic optimal control models and also in our presentations. We introduce the following notation.

Definition 2.1.3 (Extremal Operators)**.** *The extremal operators* $\mathcal{M}^{\pm}_{\mathcal{L}}$ *with respect to a family* $\mathcal{L} \subseteq \mathcal{L}_0$ *are defined by,*

$$\begin{aligned}
\mathcal{M}^{-}_{\mathcal{L}} u(x,t) &:= \inf_{(K,b)\in\mathcal{L}} L_{K,b}u(x,t),\\
\mathcal{M}^{+}_{\mathcal{L}} u(x,t) &:= \sup_{(K,b)\in\mathcal{L}} L_{K,b}u(x,t).
\end{aligned}$$

Whenever $\mathcal{L} = \mathcal{K} \times \{0\}$ we also denote $\mathcal{M}^{\pm}_{\mathcal{L}} = \mathcal{M}^{\pm}_{\mathcal{K}}$. For example, the ones with respect to $\mathcal{K}_0$ can be explicitly written as,

$$\begin{aligned}
\mathcal{M}^{-}_{\mathcal{K}_0} u(x,t) &= \int \frac{\lambda\chi_{B_\rho}(y)\delta^+u(x,t;y) - \Lambda\delta^-u(x,t;y)}{|y|^{n+\sigma}}dy,\\
\mathcal{M}^{+}_{\mathcal{K}_0} u(x,t) &= \int \frac{\Lambda\delta^+u(x,t;y) - \lambda\chi_{B_\rho}(y)\delta^-u(x,t;y)}{|y|^{n+\sigma}}dy.
\end{aligned}$$

where $\delta u = \delta^+ u - \delta^- u$ is the sign decomposition of δu.

The uniform ellipticity property with respect to $\mathcal{L}$ in $\Omega \times (a,b]$ will be frequently used by saying that for sufficiently smooth functions u, v and for every $(y,s) \in \Omega \times (a,b]$,

$$\mathcal{M}^{-}_{\mathcal{L}}(u-v)(y,s) \le (Iu - Iv)(y,s) \le \mathcal{M}^{+}_{\mathcal{L}}(u-v)(y,s).$$

2.1.3 Limit as $\sigma \to 2$

The following section is independent of the time variable so we omit it. Here we start using the weight ω_σ which measures the contribution of the tail to the nonlocal operator of order $\sigma \in (0,2)$,

$$\omega_\sigma(y) := \min(1, |y|^{-(n+\sigma)}).$$

It just requires a simple computation to check the following proposition.

Proposition 2.1.1. *Let $\sigma \in (0,2)$, $u \in C^{1,1}(0) \cap L^1(\omega_\sigma)$ such that,*

$$\delta u(0;y) - \operatorname{tr}(My \otimes y) \leq |y|^2 \omega(|y|) \text{ for } y \in B_1, \tag{2.1.7}$$

for some bounded $\omega : (0,1) \to \mathbb{R}^+$ and some $n \times n$ symmetric matrix M. Then, for every symmetric $n \times n$ matrix A and $K_{k,\sigma}(y) := (2-\sigma)k(y)|y|^{-(n+\sigma)}$,

$$\begin{aligned} L_{K_{k,\sigma}} u(0) - \operatorname{tr}(AM) \leq C(2-\sigma) \Bigg(\operatorname{tr} \left(M \int_{B_1} \frac{(k(y)Id - A) y \otimes y}{|y|^{n+\sigma}} dy \right) \\ + \|k\|_\infty \left(\int_0^1 \frac{\omega(r)}{r^{1-(2-\sigma)}} dr + \|u - u(0)\|_{L^1(\omega_\sigma)} \right) \Bigg), \end{aligned}$$

for some universal constant C depending only of the dimension.

For example, if as $\sigma \nearrow 2$,

$$(2-\sigma) \int_{B_1} \frac{y \otimes y}{|y|^{n+\sigma}} k(y) dy \to A_k \tag{2.1.8}$$

we get that $L_{K_{k,\sigma}} u(0) \to \frac{1}{2} \operatorname{tr}(A_k D^2 u(0))$ with a modulus of convergence depending only on ω, giving the modulus of convergence of the second order difference $\delta u(0;y) \to (1/2) \operatorname{tr}(D^2 u(0) y \otimes y)$, the modulus of convergence of (2.1.8) and $\|k\|_\infty$.

The limit (2.1.8) holds if $k(r\cdot) \to k_0$ in $L^1(\partial B_1)$ as $r \searrow 0$. Then A_k can be explicitly computed by,

$$A_k = \int_{\partial B_1} \theta \otimes \theta k_0(\theta) d\theta.$$

Let $\mathcal{K} = \{k\}$ be a set of kernels such that the limit (2.1.8) converges in a uniform way,

$$\lim_{\sigma \nearrow 2} \sup_{k \in \mathcal{K}} \left| \frac{1}{2} \int_{B_1} y \otimes y K_{k,\sigma}(y) dy - A_k \right| = 0.$$

A function $I \in C(\mathbb{R}^{\mathcal{K}})$, defines an operator I_σ of order $\sigma \in (0,2)$, by

$$I_\sigma u(y) := I((L_{K_{k,\sigma},b} u(y))_{k \in \mathcal{K}}).$$

As $\sigma \nearrow 2$ we obtain that

$$I_\sigma u(0) \to I_2 u(0) := I(((1/2)\operatorname{tr}(A_k D^2 u(0)))_{k \in \mathcal{K}}).$$

A useful example at the moment to build barriers is the limit of $\mathcal{M}^{\pm}_{\mathcal{K}_0}$.

Proposition 2.1.2. *Given $u \in C^2(0) \cap L^1(\omega_{\sigma_0})$ for some $\sigma_0 < 2$,*

$$\begin{aligned}
\lim_{\sigma \nearrow 2} \mathcal{M}^-_{\mathcal{K}_0} u(0) &= \int_{\partial B_1} \left(\left(\theta^t D^2 u(0) \theta\right)^+ \lambda - \left(\theta^t D^2 u(0) \theta\right)^- \Lambda \right) d\theta,\\
\lim_{\sigma \nearrow 2} \mathcal{M}^+_{\mathcal{K}_0} u(0) &= \int_{\partial B_1} \left(\left(\theta^t D^2 u(0) \theta\right)^+ \Lambda - \left(\theta^t D^2 u(0) \theta\right)^- \lambda \right) d\theta.
\end{aligned}$$

In particular, these second order operators are comparable to the classical extremal Pucci operators. For some universal $C \geq 1$ depending only on the dimension,

$$\begin{aligned}
\inf_{A \in [\lambda/C, C\Lambda]} \operatorname{tr}(AM) &\leq \int_{\partial B_1} \left(\left(\theta^t M \theta\right)^+ \lambda - \left(\theta^t M \theta\right)^- \Lambda \right) d\theta \leq \inf_{A \in [\lambda,\Lambda]} \operatorname{tr}(AM),\\
\sup_{A \in [\lambda,\Lambda]} \operatorname{tr}(AM) &\leq \int_{\partial B_1} \left(\left(\theta^t M \theta\right)^+ \Lambda - \left(\theta^t M \theta\right)^- \lambda \right) d\theta \leq \sup_{A \in [\lambda/C, C\Lambda]} \operatorname{tr}(AM).
\end{aligned}$$

Finally, the modulus of convergence above depends only on the modulus of convergence of $\delta u(0;y) \to (1/2)\operatorname{tr}(D^2 u(0) y \otimes y)$.

2.2 Viscosity Solutions

The set of test functions we are about to define imposes sufficient requirements in order to evaluate the previous nonlocal operator on a cylinder $C_{r,\tau}(x,t)$. First of all we need to impose some continuity for the tails.

Definition 2.2.1. *The space $LSC((a,b] \mapsto L^1(\omega_\sigma))$ consists of all measurable functions $u : \mathbb{R}^n \times (a,b] \to \mathbb{R}$ such that for every $t \in (a,b]$,*

1. $\|u(\cdot,t)^-\|_{L^1(\omega_\sigma)} < \infty$.
2. $\lim_{\tau \nearrow 0} \|(u(\cdot,t) - u(\cdot,t-\tau))^+\|_{L^1(\omega_\sigma)} = 0$.

Similarly, $u \in USC((a,b] \mapsto L^1(\omega_\sigma))$ if $-u \in LSC((a,b] \mapsto L^1(\omega_\sigma))$ and $C((a,b] \mapsto L^1(\omega_\sigma)) = LSC((a,b] \mapsto L^1(\omega_\sigma)) \cap USC((a,b] \mapsto L^1(\omega_\sigma))$.

Definition 2.2.2 (Test functions)**.** *A lower semicontinuous test function is defined as a pair $(\varphi, C_{r,\tau}(x,t))$, such that $\varphi \in C^{1,1}_x C^1_t(C_{r,\tau}(x,t)) \cap LSC((t-\tau,t] \mapsto L^1(\omega_\sigma))$.*

Similarly, $(\varphi, C_{r,\tau}(x,t))$ is an upper semicontinuous test function if the pair $(-\varphi, C_{r,\tau}(x,t))$ is a lower semicontinuous test function.

Test functions not only have enough regularity to evaluate I but also to make it semicontinuous.

Property 2.2.1. *Given $\mathcal{L} \subseteq \mathcal{K}_0 \times B_\beta$, $I \in LSC(C_{r,\tau}(x,t) \times \mathbb{R}^{\mathcal{L}})$ and a lower semicontinuous test function $(\varphi, C_{r,\tau}(x,t))$, the function $I\varphi \in LSC(C_{r,\tau}(x,t))$.*

The idea to show the semicontinuity in space or time is the same. One needs to show that $\{L_{K,b}\varphi\}_{(K,b)\in\mathcal{K}_0\times B_\beta}$ has a uniform modulus of continuity in space and a uniform modulus of semicontinuity in time.

Whenever the cylinder in the Definition 2.2.2 becomes irrelevant we will refer to the test function $(\varphi, C_{r,\tau}(x,t))$ just by φ.

Definition 2.2.3. *Given a function u and a test function φ, we say that φ **touches** u **from below at** (x,t) if,*

1. $\varphi(x,t) = u(x,t)$,

2. $\varphi(y,s) \leq u(y,s)$ *for* $(y,s) \in \mathbb{R}^n \times (t-\tau, t]$.

*Similarly, φ touches u from above at (x,t) if $-\varphi$ touches $-u$ from below at (x,t). Finally, φ **strictly touches** u **from above or below at** (x,t) if the inequality becomes strict outside of (x,t).*

Definition 2.2.4 (Viscosity Solution)**.** *Given an elliptic operator I and a function f, a function $u \in LSC(\Omega \times (a,b]) \cap LSC((a,b] \mapsto L^1(\omega_\sigma))$ is said to be a **viscosity super solution to** $u_t - Iu \geq f$ **in** $\Omega \times (a,b]$, if for every lower semicontinuous test function $(\varphi, C_{r,\tau}(x,t))$ touching u from below at $(x,t) \in \Omega \times (a,b]$, we have that $\varphi_{t^-}(x,t) - I\varphi(x,t) \geq f(x,t)$.*

In the previous definition we are using the **time derivative towards the past**, which is the appropriated one for time evolution models,

$$\varphi_{t^-}(x,t) := \lim_{\varepsilon \nearrow 0} \frac{\varphi(x,t) - \varphi(x,t-\varepsilon)}{\varepsilon}.$$

We also write that "$u_t - Iu \geq f$ **in viscosity in** $\Omega \times (a,b]$" whenever u is a viscosity super solution to $u_t - Iu \geq f$ in $\Omega \times (a,b]$.

The definition of u being a **viscosity sub solution to** $u_t - Iu \leq f$ **in** $\Omega \times (a,b]$ is done similarly to the definition of super solution replacing LSC by USC, contact from below by contact from above and reversing the last inequality. Again, we say that "$u_t - Iu \leq f$ **in viscosity in** $\Omega \times (a,b]$" whenever u is a viscosity sub solution to $u_t - Iu \leq f$ in $\Omega \times (a,b]$.

Finally, a **viscosity solution to** $u_t - Iu = f$ **in** $\Omega \times (a,b]$ is one which is simuoltaneously a viscosity super and sub solution. We also denote it by saying that "$u_t - Iu = f$ **in viscosity in** $\Omega \times (a,b]$".

The requirement for the functions to be in $LSC((a,b] \mapsto L^1(\omega_s))$ or $USC((a,b] \mapsto L^1(\omega_s))$ can be illustrated by the following example. Consider the fractional heat equation $u_t - \Delta^{\sigma/2}u = 0$ in $\Omega \times (a,b]$ with initial and boundary data equal to zero. Clearly, it is solved classically by u being identically zero in $\mathbb{R}^n \times [a,b]$. By modifying u at $\Omega^c \times \{b\}$, we obtain that u still solves the equation in $\Omega \times (a,b)$ but fails at the last time as any modification done to the boundary data will immediately modify $\Delta^{\sigma/2}u$. This implies that, for such new boundary and initial data, there will be no classical solution of the fractional heat equation.

Remark 2.2.1. *Given* $u \in LSC((t-\tau,t] \mapsto L^1(\omega_\sigma)) \cup USC((t-\tau,t] \mapsto L^1(\omega_\sigma))$ *and* $\varphi \in C^{1,1}_x C^1_t(C_{r,\tau}(x,t))$ *we define the test function* $(\varphi_u, C_{r,\tau}(x,t))$

as

$$\varphi_u = \begin{cases} \varphi \text{ in } C_{r,\tau}(x,t), \\ u \text{ in } \partial_p C_{r,\tau}(x,t). \end{cases}$$

This is a admissible test function to test against u when we only have a smooth function touching u in a small neighborhood of the contact point.

Remark 2.2.2. *Sometimes we need to allow some room in the estimations by assuming that φ strictly touches u at (x,t). This could be treated by adding or subtracting the following perturbation $\psi_{\delta,x,t}$ to φ*

$$\psi_{\delta,x,t}(y,s) = \delta \begin{cases} (|y-x|^2 - (s-t)) & \text{for } (y,s) \in C_{r,\tau}(x,t), \\ 1 & \text{for } (y,s) \in \partial_p C_{r,\tau}(x,t). \end{cases}$$

Given that I is (semi)continuous, this small perturbation adds an error that can be send to zero at the end of the proof. For this reason, we will assume sometimes that a test function strictly touches the viscosity sub or super solution.

Property 2.2.2. *Let I, J be elliptic operators and u satisfies,*

$$u_t - Iu \geq f \text{ in viscosity in } \Omega \times (a,b],$$

Then:

1. *Given that v satisfies*

$$v_t - Iv \geq g \text{ in viscosity in } \Omega \times (a,b],$$

then for $w = \min(u,v)$ and $h = f\chi_{u<v} + g\chi_{v<u} + \max(f,g)\chi_{u=v}$ we also have that,

$$w_t - Iw \geq h \text{ in viscosity in } \Omega \times (a,b].$$

2. Given that $I \geq J$ and $g \leq f$, then u also satisfies,

$$u_t - Ju \geq g \text{ in viscosity in } \Omega \times (a, b].$$

3. Given that I is uniformly elliptic and $(\varphi, C_{r,\tau}(x,t))$ is a lower semicontinuous test function touching u from below at some point (x,t), then the following quantities are well defined for $p = D\varphi(x,t)$,

$$L^p_{K,b}u(x,t) := \lim_{\varepsilon \to 0} \int_{B^c_\varepsilon} \delta^p u(x,t;y) K(y) dy + b \cdot p,$$
$$\delta^p u(x,t;y) := u(x+y,t) - u(x,t) - p \cdot y \chi_{B_1}(y),$$

and satisfies

$$\varphi_{t^-}(x,t) - I\left(x, t, L^p_{K,b}u(x,t)\right) \geq f(x,t).$$

The first two properties are immediate from the definition. The idea of the proof for the last one is to test u with a family of test functions that incorporates the values of u and closes the principal value of the integral, namely $\{(\varphi_{u,\varepsilon}, C_{\varepsilon,\tau}(x,t))\}_{\varepsilon \leq r}$ where $\varphi_{u,\varepsilon}$ are defined as in the Remark 2.2.1.

There will be two ways to control the convergence of the integrals, one coming from the equation and the other by the contact from below by a regular function. The uniform ellipticity is used to control the errors by the Lipschitz modulus of continuity of $I(x,t,\cdot)$. See [30] for complete details.

A consequence of the previous property is the fact that classical solution are also solutions in the viscosity sense.

Property 2.2.3. *Let I be a uniformly elliptic operator, f be a continuous function and $u \in C^{1,1-\sigma}_x C^1_t(\Omega \times (a,b]) \cap USC((a,b] \mapsto L^1(\omega_\sigma))$ such that,*

$$u_t - Iu \leq f \text{ classically in } \Omega \times (a,b].$$

Then it also holds that,

$$u_t - Iu \leq f \text{ in viscosity in } \Omega \times (a,b].$$

The idea is to show that the set of points where u can be touched from above is in fact dense, therefore the equations holds in such a dense set. The regularity of u then implies that the equation has to hold at every point. Again, see [30] for further details.

Chapter 3

Qualitative Properties

The qualitative properties we treat on this chapter are the Stability of the equations by Γ-convergence, the Maximum Principle, the uniform ellipticity identity for viscosity solutions and the Comparison Principle. All of them lead us to the existence and uniqueness of viscosity solutions by Perron's method provided that we have barriers that force the solution to take the boundary and initial values in a continuous way. These barriers are also constructed in this chapter and, moreover, this is done in such a way that they will allow us to get regularity estimates up to the boundary on Chapter 5.

3.1 Stability

The following definition of Γ convergence can be understood as uniform convergence from below in the domain $\Omega \times (a, b]$, which is the classical notion also appearing in the second order theory of viscosity solutions, plus some control in $L^1(\omega_\sigma)$ for the tails. Whenever u_i and $-u_i$ both converge in the Γ sense to u and $-u$ respectively, then we recover that u_i converges locally uniformly to u in $\Omega \times (a, b]$ and also locally uniformly in $C((a, b] \mapsto L^1(\omega_\sigma))$.

Definition 3.1.1 (Γ-convergence). *A sequence of functions* $\{u_i\}_{i\in\mathbb{N}} \subseteq LSC(\Omega\times$

$(a,b]) \cap LSC((a,b] \mapsto L^1(\omega_\sigma))$ *Γ-converges to a function u if:*

1. *For every sequence $(x_i,t_i) \to (x,t^-) \in \Omega \times (a,b]$, $\liminf_{i\to\infty} u_i(x_i,t_i) \geq u(x,t)$,*

2. *For every sequence $t_i \to t^- \in (a,b]$, $\|(u(\cdot,t) - u(\cdot,t_i))^+\|_{L^1(\omega_\sigma)} \to 0$,*

3. *For every $(x,t) \in \Omega \times (a,b]$, there exists a sequence $(x_i,t_i) \to (x,t^-)$ such that $u_i(x_i,t_i) \to u(x,t)$,*

The previous definition implies that the limit u also belongs to $LSC(\Omega \times (a,b]) \cap LSC((a,b] \mapsto L^1(\omega_\sigma))$. Another important property of Γ-convergence is the following one. If $u_i \to u$ in the Γ sense in $\Omega \times (a,b]$ and u has a strict local minimum at some $(x,t) \in \Omega \times (a,b]$ then there exists a sequence $(x_i,t_i) \to (x,t^-)$ such that u_i has a strict local minimum at (x_i,t_i).

We can use this last property whenever we are given a test function φ, strictly touching u from below at (x,t) in $C_{r,\tau}(x,t)$. Then by a vertical translation we get test functions $(\varphi + d_i)$ touching u_i from below at (x_i,t_i) in $C_{r,\tau}(x,t)$ such that $d_i \to 0$ and $(x_i,t_i) \to (x,t^-)$.

Theorem 3.1.1 (Stability)**.** *Let I be a lower semicontinuous elliptic operator and $\{u_i\}_{i\geq 1}$ and $\{f_i\}_{i\geq 1}$ be sequences of functions such that:*

1. *$(u_i)_t - Iu_i \geq f_i$ in the viscosity sense in $\Omega \times (a,b]$,*

2. *$u_i \to u$ in the Γ sense in $\Omega \times (a,b]$,*

3. $\liminf_{i\to\infty} f_i(x_i,t_i) \geq f(x,t)$ *for every* $(x_i,t_i) \to (x,t^-)$ *in* $\Omega \times (a,b]$*,*

Then

$$u_t - Iu \geq f \text{ in viscosity in } \Omega \times (a,b].$$

Proof. Let φ be a test function touching u from above at (x,t). We need to show that $\varphi_t(x,t) - I\varphi(x,t) \geq f(x,t)$. By Remark 2.2.2 we can assume, without loss of generality, that the contact is strict.

By Γ-convergence, there exists sequences $(x_i,t_i) \to (x,t^-)$ in $C_{r,\tau}(x,t)$ and $d_i \to 0$ such that $(\varphi + d_i)$ strictly touches u_i from below at (x_i,t_i) but just in $C_{r,\tau}(x,t)$. So far, $(\varphi + d_i)$ is not an admissible test function for $(u_i)_t - Iu_i \geq f_i$ but we can modify it as in the Remark 2.2.1, to obtain the admissible test function $\varphi_i = (\varphi + d_i)_{u_i}$.

From the hypothesis we get immediately that $(\varphi_i)_t(x_i,t_i) = \varphi_t(x_i,t_i) \to \varphi_t(x,t)$ as $i \to \infty$ meanwhile $\liminf_{i\to\infty} f_i(x_i,t_i) \geq f(x,t)$. The lower semicontinuity of I and the second property of Γ convergence, controlling the tails of u_i, imply that $\liminf_{i\to\infty} I\varphi_i(x_i,t_i) \geq I\varphi(x,t)$ which concludes the proof. $\square$

Remark 3.1.2. *The ellipticity of I was not really used in this proof but just lower semicontinuity. A more general stability result will be discussed on Theorem 6.0.6 for operators which are not necessarily elliptic.*

3.2 Comparison Principle

The Comparison Principle for elliptic equations says that whenever u and v are sub and super solutions of the same equation such that $u \leq v$ in the parabolic boundary of the domain then the ordering gets also preserved inside the domain. This implies immediately the uniqueness of solutions. The first step is to prove the Maximum Principle which is the Comparison Principle when one of the functions is identically zero. After this the Comparison Principle would be immediate if we can use the uniform ellipticity identity for viscosity solutions. However, this is not directly implied from the definitions. We prove that this identity holds for translation invariant operators by using the classical sup-convolution which regularizes the solution meanwhile preserving the equation.

3.2.1 Maximum Principle

Theorem 3.2.1 (Maximum Principle)**.** *Let I be a uniformly elliptic operator and w a function such that,*

$$w_t - Iw \leq f \text{ in viscosity in } \Omega \times (a,b].$$

Then

$$\sup_{\Omega\times(a,b]} w \leq \sup_{\partial_p(\Omega\times(a,b])} w + C\|(f - I0)^+\|_\infty,$$

for some universal constant $C > 0$ depending on Ω but independent of $\sigma \in [1,2)$.

Proof. Assume without loss of generality that $\sup_{\partial_p(\Omega\times(a,b])} w = 0$ and $I0 = 0$. Otherwise, apply the following proof to $(w - \sup_{\partial_p(\Omega\times(a,b]} w)$ and $f - I0$ (recall also that from our definition of viscosity sub solution $\sup_{\Omega\times(a,b]} u < \infty$).

We will use a rescaling of $\psi(y) = (2 - |y|^2)\chi_{B_1}(y)$ as a test function for w. The important thing to notice is that $\mathcal{M}^+_{\mathcal{L}_0}\psi \le -\delta_0$ in some ball B_{δ_1} for some universal constants $\delta_0 > 0$ and $\delta_1 \in (0,1)$ independent of $\sigma \in [1,2)$ (however going to zero as $\rho \to 0$, coming from $\mathcal{L}_0 = \mathcal{L}_0(\sigma,\rho,\lambda,\Lambda)$). It can be proved by using Proposition 2.1.2 because, for each $\sigma \in [1,2)$, $\mathcal{M}^+_{\mathcal{L}_0}\psi$ is strictly negative in a neighborhood of the origin and neither this negative quantity or the neighborhood degenerate in the limit when σ goes to two.

Assume that $M := \sup_{\Omega\times(a,b]} u \ge 0$, otherwise there is nothing to prove. Assume also that $\Omega \subseteq B_R$, for $R = \mathrm{diam}(\Omega)$, by choosing an appropriated system of coordinates. By the assumptions, $\varphi := \alpha\psi(\cdot/R)$ gives us a test function touching u from above at $(x,t) \in \Omega \times (a,b]$ for some $\alpha \in [M/2, M]$. Then we should have that

$$\begin{aligned}\|f^+\|_\infty \ge f(x,t) \ge \varphi_t(x,t) - I\varphi(x,t) &\ge -\mathcal{M}^+_{\mathcal{L}_0}\varphi(x)\\ &\ge (M/2)(R\delta_1^{-1})^{-\sigma}\delta_0,\end{aligned}$$

giving us the desired bound. $\square$

As a consequence we obtain uniqueness of the zero solution for,

$$\begin{aligned} u_t - Iu = I0 \text{ in viscosity in } \Omega \times (a,b],\\ u = 0 \quad \text{in } \partial_p(\Omega \times (a,b]). \end{aligned}$$

Another Corollary allows us to get a bound for w only assuming that w is bounded around $\Omega \times (a,b]$. It is proved by truncating u away from the domain $\Omega \times (a,b]$.

Corollary 3.2.2. *Let I be a uniformly elliptic operator and w a function such that,*

$$w_t - Iw \leq f \text{ in viscosity in } \Omega \times (a,b].$$

Then, for $\Omega \subseteq B_{R/2}$,

$$\sup_{\Omega \times (a,b]} w \leq \sup_{C_{R,b-a}(0,b) \setminus (\Omega \times (a,b])} w + C \left(\|(f - I0)^+\|_\infty + \|w^+ \chi_{B_R^c}\|_{L^\infty((a,b] \mapsto L^1(\omega_\sigma))} \right),$$

for some constant $C > 0$ depending on Ω but independent of $\sigma \in [1,2)$ and R.

Notice that as R becomes larger and larger, $\sup_{C_{R,b-a}(0,b) \setminus (\Omega \times (a,b])} w$ becomes larger and $\|w^+ \chi_{B_R^c}\|_{L^\infty((a,b] \mapsto L^1(\omega_\sigma))}$ becomes smaller. In some cases it is possible to find an optimal R for the previous estimate. For example, if $\Omega \times (a,b] = C_{1,1}$ and $w \chi_{B_1^c} \leq M|y|^{n+\sigma_0}$ for some $\sigma_0 < \sigma$ then $\sup_{C_{1,1}} w \leq C((\sigma - \sigma_0)^{-1} M + \|(f - I0)^+\|_\infty)$.

3.2.2 Uniform Ellipticity Identity for Viscosity Solutions

Theorem 3.2.3 (Uniform ellipticity identity for viscosity solutions)**.** *Let I be a translation invariant, uniformly elliptic operator with respect to $\mathfrak{L}$, $f, -g \in USC(\Omega \times (a,b])$ and u and v such that,*

$$u_t - Iu \leq f \text{ in viscosity in } \Omega \times (a,b],$$

$$v_t - Iv \geq g \text{ in viscosity in } \Omega \times (a,b].$$

Then for $w = u - v$,

$$w_t - \mathcal{M}^+_{\mathcal{L}} w \leq f - g \text{ in viscosity in } \Omega \times (a, b].$$

The previous Theorem is just the uniform ellipticity of I if at least one of the functions u or v is smooth. Whenever u and v are not smooth it is not immediate how to pass this identity to viscosity solution. The idea is then to regularize u and v in a suitable way that preserves the equation.

Definition 3.2.1 (Sup-convolution)**.** *Let $a' \in (a, b)$ and $u \in USC(\mathbb{R}^n \times (a, b]) \cap L^\infty(\mathbb{R}^n \times (a, b])$. We define the upper ε-envelope $u^\varepsilon : \mathbb{R}^n \times (a', b] \to \mathbb{R}$ as*

$$u^\varepsilon(x, t) = \sup_{(y,s) \in \mathbb{R}^n \times [a', t]} \left(u(y, s) - \varepsilon^{-1} P(y - x, s - t) \right).$$

For $P(y, s) = (|y|^2 - s)$.

Similarly we define the lower ε-envelope for $v \in LSC(\mathbb{R}^n \times (a, b]) \cap L^\infty(\mathbb{R}^n \times (a, b])$ by $v_\varepsilon = -(-v)^\varepsilon$.

Notice that the supremum above is taken for $s \leq t$ which is the right notion coming from evolution equations.

Geometrically, given $(x, t) \in \mathbb{R}^n \times (a, b]$, $u - \varepsilon^{-1} P(\cdot - x, \cdot - t)$ measures the vertical distance between u and the convex parabolic paraboloid $\varepsilon^{-1} P(\cdot - x, \cdot - t)$. By taking the supremum we are computing how much do we need to translate $\varepsilon^{-1} P(\cdot - x, \cdot - t)$ vertically in order to make the graphs touch each other at some point $(x^*, t^*) \in \bar{\mathbb{R}}^n \times [a', t]$. See figure 3.1.

The following properties can be proved by duality arguments as in [17].

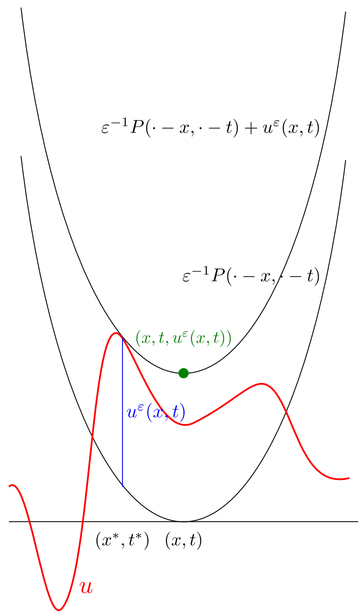


Figure 3.1: Geometric interpretation of the sup-convolution.

Property 3.2.1. *Let u^ε be the upper ε-envelope for u and for $(x,t) \in \mathbb{R}^n \times (a', b]$ let $(x^*, t^*) \in \mathbb{R}^n \times [a', t]$ such that*

$$P(x^* - x, t^* - t) = \varepsilon(u(x^*, t^*) - u^\varepsilon(x,t)).$$

Then,

1. $-u^\varepsilon \nearrow -u$ *in the* Γ *sense as* $\varepsilon \to 0$.

2. u^ε *is* $C^{1,1}(\mathbb{R}^n \times (a', b])$ *from below in the parabolic sense, meaning that for every* $(x,t) \in \mathbb{R}^n \times (a', b]$ *the paraboloid* $u(x^*, t^*) - \varepsilon^{-1}P^*(\cdot - x^*, \cdot - t^*)$, *for* $P^*(y,s) = (|y|^2 + s)$, *touches* u^ε *from below and towards the past at*

(x, t),

$$u^\varepsilon(x,t) = u(x^*,t^*) - \varepsilon^{-1}P^*(x - x^*, x - t^*),$$

$$u^\varepsilon(y,s) \geq u(x^*,t^*) - \varepsilon^{-1}P^*(y - x^*, s - t^*) \text{ for } (y,s) \in \mathbb{R}^n \times (a', t].$$

Remark 3.2.4. *The last property tells us that for every $t \in (a', b]$, $u^\varepsilon(\cdot, s)$ is semi-convex. By a result of Alexandrov we then know that it is also twice differentiable a.e.. About the regularity in time, we know that for every $x \in \mathbb{R}^n$, the function $s \mapsto u^\varepsilon(x,s) + \varepsilon s$ is nondecreasing, which implies that $u_t^\varepsilon(x, \cdot)$ is also well defined a.e.*

Lemma 3.2.5. *Let I be a translation invariant, uniformly elliptic operator with respect to $\mathcal{L}$, $f \in USC(\Omega \times (a,b])$ and $u \in L^\infty(\mathbb{R}^n \times (a,b])$ satisfies,*

$$u_t - Iu \leq f \text{ in viscosity in } \Omega \times (a,b].$$

Then, for $\Omega' \subset\subset \Omega$, $a'' \in (a', b)$ and ε small enough u^ε also satisfies

$$u_t^\varepsilon - Iu^\varepsilon \leq f + \omega(\varepsilon) \text{ in viscosity in } \Omega' \times (a'', b],$$

for some $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Consider a test function $(\varphi, C_{r,\tau}(x,t))$ touching u^ε from above at $(x,t) \in \Omega' \times (a'', b]$. We need to show that $\varphi_{t^-}(x,t) - I\varphi(x,t) \leq f(x,t) + \omega(\varepsilon)$. By translating the graph of φ from $(x, t, u^\varepsilon(x,t))$ to $(x^*, t^*, u(x^*, t^*))$ we obtain a test function for u at (x^*, t^*). Indeed, let

$$\varphi^*(y,s) = \varphi(y - (x^* - x), s - (t^* - t)) + (u(x^*,t^*) - u^\varepsilon(x,t))$$

Clearly $\varphi^*(x^*,t^*) = u(x^*,t^*)$ and, for $(y,s) \in \mathbb{R}^n \times (t^* - \tau, t^*]$

$$\begin{aligned}
\varphi^*(y,s) &\geq u^\varepsilon(y - (x^* - x), s - (t^* - t)) + (u(x^*,t^*) - u^\varepsilon(x,t)),\\
&= u^\varepsilon(y - (x^* - x), s - (t^* - t)) + \varepsilon^{-1}P(x^* - x, t^* - t),\\
&\geq u(y,s)
\end{aligned}$$

Now we need to make sure that $(x^*,t^*) \in \Omega \times (a,b]$ before plugging φ^* into the equation.

We control that (x^*,t^*) is not too far from (x,t) by

$$|x - x^*|^2 + (t - t^*) = \varepsilon(u(x^*,t^*) - u^\varepsilon(x,t)) \leq 2\varepsilon\|u\|_\infty.$$

Given that $(x,t) \in \Omega' \times (a'',b]$ we can make $(x^*,t^*) \in \Omega \times (a,b]$ by choosing ε sufficiently small. Then,

$$\begin{aligned}
\varphi^*_t(x^*,t^*) - I\varphi^*(x^*,t^*) &= \varphi_t(x,t) - I(x^*,t^*,D\varphi(x,t),(L_K\varphi(x,t))_{K\in\mathcal{K}_0}),\\
&\leq f(x^*,t^*) + (f(x,t) - f(x^*,t^*)),\\
&\leq f(x^*,t^*) + \omega(\varepsilon)
\end{aligned}$$

where $\omega(\varepsilon)$ is some modulus of upper semicontinuity of f in $C_{(2\varepsilon\|u\|_\infty)^{1/2},2\varepsilon\|u\|_\infty}(x,t)$.

□

Remark 3.2.6. *The previous result actually did not use the ellipticity of I but only the Lipschitz modulus of continuity implied by it. The result would also hold for I depending on $(y,s) \in \Omega \times (a,b]$ whenever we can control the oscillation of* $\sup_{(l_{K,b})\in\mathbb{R}^{\mathcal{L}}} I(\cdot,\cdot,l_{K,b})$.

Lemma 3.2.7. *Let I be a translation invariant, uniformly elliptic operator with respect to $\mathcal{L}$, $f, -g \in USC(\Omega \times (a,b])$, $u, -v \in USC(\mathbb{R}^n \times (a,b]) \cap L^\infty(\mathbb{R}^n \times (a,b])$ functions such that,*

$$u_t - Iu \leq f \text{ in viscosity in } \Omega \times (a,b],$$

$$v_t - Iv \geq g \text{ in viscosity in } \Omega \times (a,b].$$

Then for $w = u - v$,

$$w_t - \mathcal{M}^+_{\mathcal{L}} w \leq f - g \text{ in viscosity in } \Omega \times (a,b].$$

In [10] the proof of the analogous Lemma was simplified by the Lemma 4.3, analogous to the last property we have in 2.2.2. In this case we do not have control any more of the set where the time derivative can be evaluated for $w^\varepsilon := (u^\varepsilon - v_\varepsilon)$ so we have to go back to the classical proof using the parabolic convex envelope or Jensen's Lemma.

Proof. We will show that for $w^\varepsilon := (u^\varepsilon - v_\varepsilon)$,

$$w^\varepsilon_t - \mathcal{M}^+_{\mathcal{L}} w^\varepsilon \leq f - g + \omega(\varepsilon) \text{ in viscosity in } \Omega \times (a,b],$$

for some $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. The result for w then follows from the stability of the equations, Theorem 3.1.1.

Let $(\varphi, C_{r,\tau}(x,t))$ a test function touching w^ε from above at $(x,t) \in \Omega \times (a,b]$ and assume without loss of generality that:

1. $C_{r,\tau}(x,t) \subset\subset \Omega \times (a,b]$,

2. $\inf_{C_{r',\tau}(x,t)}(\varphi - w^{\varepsilon}) > 0$ for every $r' \leq r$.

We need to show that,

$$\varphi_{t^-}(x,t) - \mathcal{M}^+_{\mathcal{L}}\varphi(x,t) \leq (f-g)(x,t) + \omega(\varepsilon).$$

Fix $\kappa \in (0,1)$. By subtracting a small number $\delta > 0$ we have that $\psi_\kappa = (\varphi - w^{\varepsilon} - \delta)$ still satisfies $\psi_\kappa \geq 0$ in $C_{\kappa r,\kappa\tau}(x,t)$ but now has a strictly negative minimum $-\delta$ at (x,t). Let

$$\begin{aligned}\Sigma_\kappa = &\{(y,s) \in C_{\kappa r,\kappa\tau}(x,t) : \exists p \in \mathbb{R}^n \text{ such that} \\ &\psi_\kappa(z,\varsigma) \geq p \cdot (z-y) + \psi_\kappa(y,s)\ \forall (z,\varsigma) \in C_{\kappa r, s-\kappa\tau}(x,s)\}.\end{aligned}$$

We know by a result due to K. Tso [37] that Σ_κ has positive measure. This will be rewieved in Section 4.1. In fact, we are also using here that the parabolic convex envelope of ψ_κ is $C^{1,1}$ (in the parabolic sense) which is inherited from the parabolic convexity and the fact that w^{ε} is $C^{1,1}$ from below, see Lemma 3.5 in [17].

Recall now Remark 3.2.4 which says that $u^{\varepsilon}, v_{\varepsilon} \in C^{1,1}_x C^1_t(y,s)$ for a.e. $(y,s) \in C_{\kappa r,\kappa\tau}(x,t)$. We obtain in this way $(x_\kappa, t_\kappa) \in \Sigma_\kappa$ such that $u^{\varepsilon}, v_{\varepsilon} \in C^{1,1}_x C^1_t(x_\kappa, t_\kappa)$. Given that ψ_κ has a supporting plane from below at (x_κ, t_κ) we get that,

$$(\psi_\kappa)_{t^-}(x_\kappa, t_\kappa) - \mathcal{M}^+_{\mathcal{L}}\psi_\kappa(x_\kappa, t_\kappa) \geq 0.$$

By uniform ellipticity and Lemma 3.2.5,

$$\begin{aligned}
\varphi_t(x_\kappa,t_\kappa) - \mathcal{M}^+_{\mathcal{L}}\varphi(x_\kappa,t_\kappa) &\leq w^\varepsilon_t(x_\kappa,t_\kappa) - \mathcal{M}^+_{\mathcal{L}} w^\varepsilon(x_\kappa,t_\kappa)\\
&\leq \big(u^\varepsilon_t(x_\kappa,t_\kappa) - \mathcal{M}^+_{\mathcal{L}} u^\varepsilon(x_\kappa,t_\kappa)\big)\\
&- \big((v_\varepsilon)_t(x_\kappa,t_\kappa) - \mathcal{M}^+_{\mathcal{L}} v_\varepsilon(x_\kappa,t_\kappa)\big)\\
&\leq (f-g)(x_\kappa,t_\kappa) + \omega(\varepsilon),
\end{aligned}$$

we can conclude by sending $\kappa \to 0$. □

The proof of Theorem 3.2.3 can be now recovered by mollifying and truncating u and v outside of $\Omega \times (a,b]$. As a consequence we obtain the following Comparison Principles.

Theorem 3.2.8. *Let I be a translation invariant, uniformly elliptic operator, $f, -g \in USC(\Omega \times (a,b])$ and u and v functions such that,*

$$\begin{aligned}
u_t - Iu \leq f \text{ in viscosity in } \Omega \times (a,b],\\
v_t - Iv \geq g \text{ in viscosity in } \Omega \times (a,b].
\end{aligned}$$

Then for $w = u - v$,

$$\sup_{\Omega\times(a,b]} w \leq \sup_{\partial_p(\Omega\times(a,b])} w + C\|(f-g)^+\|_\infty,$$

for some universal constant $C > 0$ depending on Ω but independent of $\sigma \in [1,2)$.

Corollary 3.2.9. *Let I be a translation invariant, uniformly elliptic operator, $f, g \in USC(\Omega \times (a,b])$ and u and v functions such that,*

$$u_t - Iu \le f \text{ in viscosity in } \Omega \times (a,b],$$
$$v_t - Iv \ge g \text{ in viscosity in } \Omega \times (a,b].$$

Then for $w = u - v$, such that $\Omega \subseteq B_{R/2}$,

$$\sup_{\Omega\times(a,b]} w \le \sup_{C_{R,b-a}(0,b)\setminus(\Omega\times(a,b])} w + C\left(\|(f-g)^+\|_\infty + \|w^+\chi_{B_R^c}\|_{L^\infty((a,b]\mapsto L^1(\omega_\sigma))}\right),$$

for some constant $C > 0$ depending on Ω but independent of $\sigma \in [1,2)$.

3.3 Existence and Uniqueness of Viscosity Solutions

We start this section giving the construction of some barriers. Then we will use Perron's method to see that the Dirichlet problem has a unique solution. The barriers we construct here will be also used at the end of Chapter 5 to get Hölder estimates up to the boundary.

3.3.1 Barriers

Lemma 3.3.1. *For $\sigma \in [1,2)$, there exists a non negative function $\psi : \mathbb{R}^n \times (-\infty, 0] \to [0,1]$ such that for some universal $\kappa, r_0 > 0$ independent of σ, and*

$$A = \{(y,s) \in B_{1+r_0} \times (-2\kappa^{-1}, 0] : 1 < |y| \le 1 + (\kappa/2)r_0(2\kappa^{-1} - t)\}$$

$$\psi_t - \mathcal{M}^+_{\mathcal{L}_0}\psi > \kappa/2 \text{ in viscosity in } A,$$
$$\psi = 0 \quad \text{in } B_1 \times \{0\},$$
$$\psi = 1 \quad \text{in } (\mathbb{R}^n \times (-\infty, 0]) \setminus A,$$

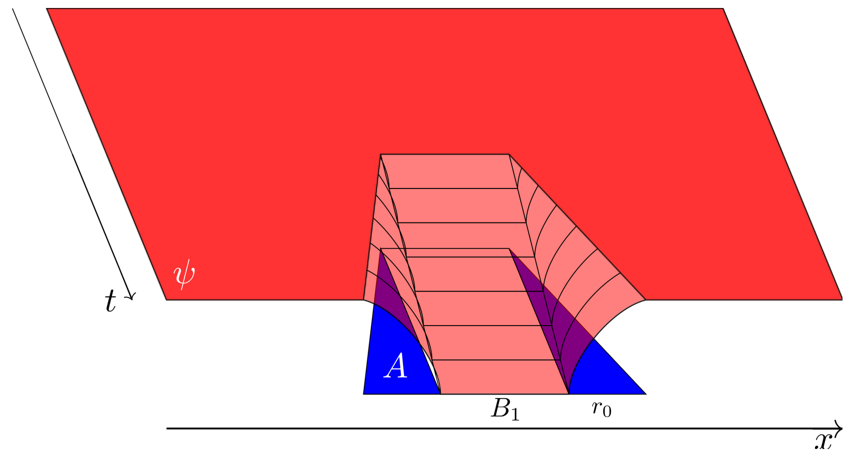


Figure 3.2: Barrier ψ.

Proof. Let

$$\varphi(y) = ((|y| - 1)^+)^\alpha.$$

We will show first that, for $\alpha, r_0 \in (0,1)$ sufficiently small, $\mathcal{M}^+_{\mathcal{L}_0}\varphi < -\kappa$ in $\bar{B}_{1+r_0} \setminus B_1$ for some universal $\kappa > 0$.

By radial symmetry it is enough to show the identity for $x = (1+r)e_1$ with $r \in (0, r_0])$. Let $x_0 = (1+r_0)e_1$, by scaling the graph of φ, centered at e_1 and sending $(x, \varphi(x))$ to $(x_0, \varphi(x_0))$ we also see that we can reduce the computation to x_0. To be more specific let $\rho = r/r_0 \in (0,1]$ and,

$$\tilde{\varphi}(y) = \rho^{-\alpha}\varphi\left(\rho(y - e_1) + e_1\right).$$

It verifies that $\tilde{\varphi} \le \varphi$ and $\tilde{\varphi}((1+|y|)e_1) = \varphi((1+|y|)e_1)$. See figure 3.3.

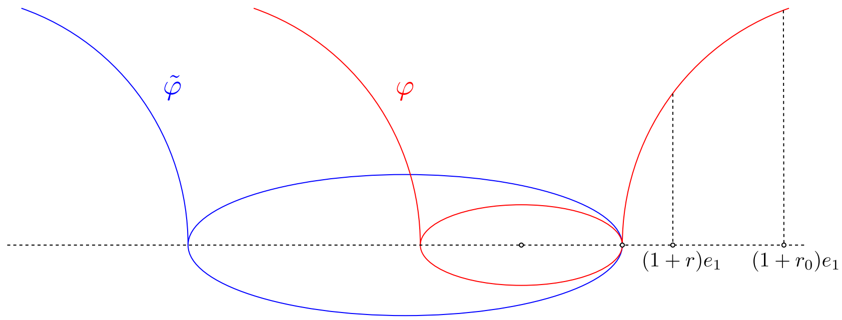


Figure 3.3: Scaling of the barrier.

Given $(K, b) \in \mathcal{L}_0$ and

$$\tilde{K}(y) = \rho^{n+\sigma} K(\rho y) \in \mathcal{K}_0',$$
$$\tilde{b} = \rho^{\sigma-1}\left(b + \int_{B_1 \setminus B_\rho} yK(y)dy\right).$$

Then,

$$L_{K,b}\varphi(x) = \rho^{\alpha-\sigma} L_{\tilde{K},\tilde{b}}\tilde{\varphi}(x_0) \leq \rho^{\alpha-\sigma} L_{\tilde{K},\tilde{b}}\varphi(x_0) \leq \rho^{\alpha-\sigma}\mathcal{M}^+_{\mathcal{L}_0}\varphi(x_0).$$

By taking the supremum on the left hand side over $(K, b) \in \mathcal{L}_0$ we conclude that $\mathcal{M}^+_{\mathcal{L}_0}\varphi(x_0) < -\kappa$ implies $\mathcal{M}^+_{\mathcal{L}_0}\varphi(x) < -\kappa$.

When $\sigma \nearrow 2$ and $\alpha \in (0, 1/2)$,

$$\begin{aligned}
\mathcal{M}^+_{\mathcal{L}_0}\varphi(x_0) &\to \int_{\partial B_1} \left(\left(\theta^t D^2\varphi(x_0)\theta\right)^+ \lambda - \left(\theta^t D^2\varphi(x_0)\theta\right)^- \Lambda\right) d\theta + \beta|D\varphi(x_0)|,\\
&\leq \sup_{A \in [\lambda/C, C\Lambda]} \operatorname{tr}(AD^2\varphi(x_0)) + \beta|D\varphi(x_0)|,\\
&= \alpha r_0^{\alpha-2}\left(\frac{\lambda(\alpha-1)}{C} + \left(C\Lambda\frac{(n-1)}{r_0+1} + \beta\right) r_0\right),\\
&< \alpha r_0^{\alpha-2}\left(-\frac{\lambda}{2C} + \left(C\Lambda(n-1) + \beta\right) r_0\right)
\end{aligned}$$

The previous quantity can be made smaller than

$$-\kappa = \frac{\alpha r_0^{\alpha-2}\lambda}{4C} < 0,$$

for r_0 sufficiently small, independently of how small is α. By the Proposition 2.1.2, we get that $\mathcal{M}^+_{\mathcal{L}_0}\varphi(x_0) < -\kappa$ for $\sigma \in [\sigma_0, 2)$ close to two.

As $\alpha \searrow 0$, $\varphi \to \chi_{B_1^c}$ for which, $\mathcal{M}^+_{\mathcal{L}_0}\chi_{B_1^c}(x_0) \to -\infty$ as $r_0 \searrow 0$ uniformly for $\sigma \in [1, \sigma_0)$ away from two. For r_0 sufficiently small we have that $\mathcal{M}^+_{\mathcal{L}_0}\chi_{B_1^c}(x_0) < -\kappa$. Finally we fix $\alpha \in (0, 1/2)$ sufficiently small such that also $\mathcal{M}^+_{\mathcal{L}_0}\varphi(x_0) < -\kappa$ holds for $\sigma \in [1, \sigma_0)$.

Now that we have proven that $\mathcal{M}_{\mathcal{L}_0}\varphi < -\kappa$ we define

$$\psi(y, s) = \max\left(\varphi(y) - \frac{\kappa}{2}s, 1\right).$$

$\square$

By combining the previous Lemma with the comparison principle we obtain the following Corollary.

Corollary 3.3.2. *Let $\varepsilon, \delta_x, \delta_t \in (0, 1)$, $C_0, C_1 \geq 0$, $a > 0$ and u such that,*

1. $\Omega \subseteq B_R \setminus B^c_{\delta_x/2}((\delta_x/2)e_1)$,
2. $u_t - \mathcal{M}^+_{\mathcal{L}_0}u \leq C_0$ *in viscosity in* $\Omega \times (-a, 0]$,
3. $u(0, 0) = 0$
4. $u \leq \varepsilon$ *in* $C_{\delta_x,\delta_t} \cap \partial_p(\Omega \times (-\delta_t, 0])$,

5. $u \le C_1$ *in* $\partial_p(\Omega \times (-\delta_t, 0])$.

Then, for κ, r_0 *and* ψ *as in Lemma 3.3.1 and* $\theta = \min\left(\frac{\delta_x}{2+r_0}, (\kappa\delta_t)^{1/\sigma}\right)$,

$$u(y,s) \le \varepsilon + \left(\frac{C_0\kappa}{2} + C_1\right)\theta^\sigma \psi\left(\frac{y-\theta e_1}{\theta}, \frac{s}{\theta^\sigma}\right) \text{ for } s \in (-a, 0]$$

A barrier that we can use for the initial values is actually much simpler. Consider $\beta : \mathbb{R}^n \to [0,1]$ a smooth function such that $\beta = 0$ in B_1 and $\beta = 1$ in B_2^c. We know that $\mathcal{M}^+_{\mathcal{L}_0}\beta$ is globally bounded and then,

$$\psi(y,s) = \beta(y) + (1 + \|\mathcal{M}^+_{\mathcal{L}_0}\beta\|_\infty)s,$$

satisfies,

$$\psi_t - \mathcal{M}_{\mathcal{L}_0}\psi \ge 1 \text{ in viscosity in } \mathbb{R}^n \times \mathbb{R},$$

$$\psi = 0 \text{ in } B_1 \times \{0\},$$

$$\psi \ge 1 \text{ in } B_2^c \times (-\infty, 0].$$

As a Corollary of the comparison principle we obtain.

Corollary 3.3.3. *Let* $\varepsilon, \delta_x, \delta_t \in (0,1)$, $C_0, C_1 \ge 0$, $a > 0$ *and* u *such that,*

1. $0 \in \Omega$,

2. $u_t - \mathcal{M}^+_{\mathcal{L}_0}u \le C_0$ *in viscosity in* $\Omega \times (0, a]$,

3. $u(0,0) = 0$

4. $u \le \varepsilon$ *in* $\bar{C}_{\delta_x,\delta_t}(0, \delta_t) \cap \partial_p(\Omega \times (0, a])$, *5.* $u \le C_1$ *in* $\partial_p(\Omega \times (0,a])$.

Then for $\theta = \min\left(\frac{\delta_x}{2}, \delta_t^{1/\sigma}\right)$ *and* ψ *as discussed just before this Lemma,*

$$u(y,s) \le \varepsilon + (C_0 + C_1)\,\theta^\sigma \psi\left(\frac{y}{\theta}, \frac{s}{\theta^\sigma}\right) \text{ for } s \in [0,a].$$

3.3.2 Perron's Method

Theorem 3.3.4. *Let*

1. $\Omega \subseteq B_{R/2}$ *with the exterior ball condition,*

2. I *be a translation invariant, uniformly elliptic operator.*

3. $f \in C(\Omega \times (a,b]) \cap L^\infty(\bar{\Omega} \times [a,b])$,

4. $g \in C((a,b] \to L^1(\omega_\sigma)) \cap L^\infty(C_{R/2,b-a}(0,b))$ *continuous at* $\bar{\Omega} \times [a,b]$.

Then, the Dirichlet problem,

$$u_t - Iu = f \text{ in } \Omega \times (a,b],$$
$$u = g \text{ in } \partial_p(\Omega \times (a,b]),$$

has a unique viscosity solution taking the boundary and initial values in a continuous way.

Proof. The uniqueness part follows from the Comparison Principle 3.2.9. For the existence we use the Stability Theorem 3.1.1 and the Comparison Principle 3.2.9. Therefore, Perron's method applies to show the existence of a viscosity solution u, defined as the smallest viscosity super solution above the boundary values given by g. The Dirichlet boundary problem gets solved by u provided that there exists barriers that force u to take the boundary and initial values in a continuous way. This is implied by the Corollaries 3.3.2 and 3.3.3. $\square$

Chapter 4

Alexandrov-Bakelman-Pucci-Krylov-Tso Estimates

The classical Alexandrov-Bakelman-Pucci-Krylov-Tso estimate controls that the measure of the contact set of the solution with its convex envelope is not too small by the size of negative infimum of the solution. This allows to prove the Point Estimate that leads to Hölder regularity as sketched on Chapter 1. For equations of order less than two we face several difficulties. The most relevant being that the Monge-Ampére measure of the convex envelope becomes singular in general. The idea we took from [10] was to cover the contact set with pieces where the solution detaches from the convex envelope in a controlled way. Therefore, instead of integrating, we add the contributions of all the pieces by using a covering lemma.

The key step for this procedure is established by Lemma 4.2.1. We learned from [35] where it is used to prove a Point Estimate for equations of order one. The same proof applies for any $\sigma \in [1, 2)$, however it deteriorates as σ goes to two.

4.1 Review of the Classical Approach

Let u be a smooth function satisfying,

$$u_t - \inf_{A\in[\lambda,\Lambda]} \mathrm{tr}(AD^2u) + \beta|Du| \geq f \text{ in } C_{1,1},$$
$$u \geq 0 \text{ in } \partial_p C_{1,1}.$$

The Alexandrov-Bakelman-Pucci-Krylov-Tso estimate tells us that,

$$\sup_{C_{1,1}} u^- \leq C \left(\iint_{\{u<0\}} (f^-)^{n+1} dyds \right)^{1/(n+1)}$$

For some universal constant C.

Assume $\beta = 0$ and $\sup_{C_{1,1}} u^- = -u(x_0,t_0) > 0$ for some $(x_0,t_0) \in C_{1,1}$. Let Γ be the parabolic convex envelope of $-u^-$. In other words, Γ is the largest function smaller than or equal to $-u^-$ which is convex in space and non increasing in time. A way to construct it is by bringing negative planes from the past until they hit the graph of $-u^-$, see figure 4.1. Specifically,

$$\Gamma(x,t) = \sup\{p\cdot x + h : p\cdot y + h \leq -u^-(y,s)\ \forall (y,s) \in C_{1,1+t}(0,t)\}.$$

Consider also $h(p,t)$, the Lengendre transform of Γ centered at x_0,

$$h(p,t) = \inf_{y\in B_1} \Gamma(y,t) - p\cdot(y-x_0).$$

Given that $\Gamma \in C^1$ and $p = D\Gamma(x,t)$ we get that

$$h(D\Gamma(x,t),t) = \Gamma(x,t) - D\Gamma(x,t)\cdot(x-x_0).$$

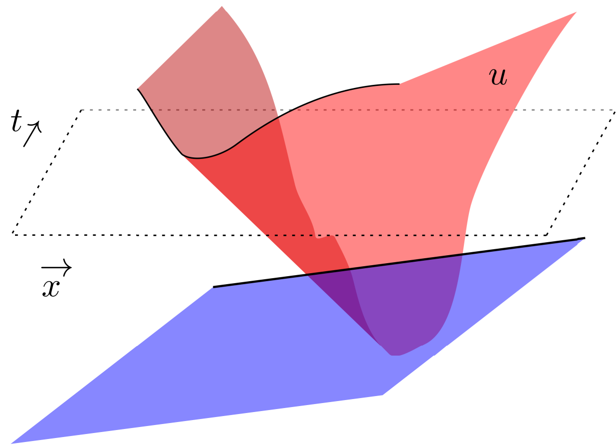


Figure 4.1: Supporting plane for the parabolic convex envelope of $-u^-$.

Geometrically, $h(x,t)$ measures the (non positive) height of the supporting plane to Γ at (x,t) measured from x_0. Finally let $\Phi(x,t) = (D\Gamma(x,t), h(x,t))$ giving us the slope and height of the supporting plane to Γ at (x,t).

The key idea of the proof is to notice that every plane intersecting the segment from $(x_0, 0)$ to $(x_0, u(x_0,t_0))$ will eventually hit the graph of u at some point in $\Sigma := \{\Gamma = u\} \subseteq \{u < 0\}$. In other words, for every pair (p,h), such that $h \in (u(x_0,t_0), 0)$ and $|p| \leq h/2$, the plane determined by (p,h) is realized as some supporting plane of Γ in Σ. Therefore

$$u^-(x_0,t_0)^{n+1} \leq C|\Phi(\Sigma)|.$$

We use the area formula to estimate the right hand side. The jacobian, which is supported in Σ, can be computed by,

$$\det(D\Phi) = -\det(D^2\Gamma)\Gamma_t \geq 0.$$

At Σ, it gets bounded by $-\det(D^2u)u_t$ which is smaller than $C(f^-)^{n+1}$ by using the equation and the AM-GM inequality. Finally we obtain the desired estimate,

$$u^-(x_0,t_0)^{n+1} \leq C\iint_{\Sigma} -\det(D^2\Gamma)\Gamma_t dyds \leq C\iint_{\{u<0\}} (f^-)^{n+1} dyds.$$

4.2 Weak Point Estimate

The following Lemma is an improvement of a weak Point Estimate that can be found in [35]. The reson we say it is weak is because as σ goes to two the estimate deterioretes and will not recover the classical Point Estimate as in [17]. Still

Lemma 4.2.1 (Key Lemma). *Let $\Delta t \in (0,1]$ and $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \text{ in viscosity in } C_{1,\Delta t}.$$

Then,

$$\inf_{C_{1/2,\Delta t/2}} u \geq \Delta t,$$

provided that for some $M > 0$,

$$\frac{|\{u > M2^{2i}\} \cap (B_{2^{i+1}} \setminus B_{2^i}) \times (-\Delta t, -\Delta t/2]|}{|(B_{2^{i+1}} \setminus B_{2^i}) \times (-\Delta t, -\Delta t/2]|} \geq M^{-1},$$

for each $i \in \{0,1,\ldots,(k-1)\}$ and $\rho \geq 2^k$ (coming from $\mathcal{K}_0 = \mathcal{K}_0(\sigma,\rho,\lambda,\Lambda)$) with $k(2-\sigma) \geq C$ for some universal constant C independent of M and $\sigma \in [1,2)$.

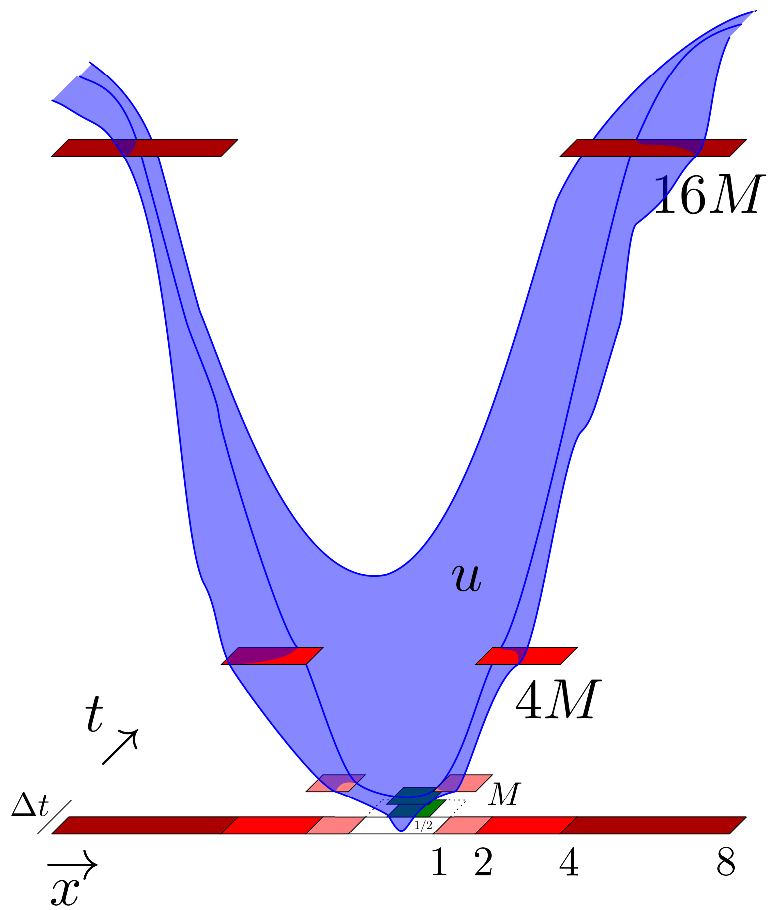


Figure 4.2: Geometric configuration for the Key Lemma.

Proof. It suffices, by the comparison principle, to show the existence of a sub solution of the same equation that remains below u on $\partial_p C_{3/4,\Delta t}$ and grows at least up to one in $C_{1/2,\Delta t/2}$. The following ansatz uses the values given by u in B_1^c acting as a forcing term that allows it to grow. Consider,

$$v(x,t) = (m(t)\varphi(x) - (t-\Delta t))\,\chi_{B_1}(x) + u(x,t)\chi_{B_1^c}(x),$$

Where φ is a smooth function taking values between zero and one with

$$\operatorname{supp}\varphi = B_{3/4},$$

$$\varphi = 1 \text{ in } B_{1/2}$$

and m is function such that $m(-\Delta t) = 0$. Notice that m determines the grow of v in $C_{1/2,1}$. To prove the Lemma we will have to arrange m such that:

1. $v_t - \mathcal{M}^-_{\mathcal{L}_0} v \leq -1$ in $C_{3/4,\Delta t}$,
2. $m \geq 2\Delta t$ in $[-\Delta t/2, 0]$.

We estimate $v_t - \mathcal{M}^-_{\mathcal{L}_0} v$ in $C_{3/4,\Delta t}$ by the uniform ellipticity identity,

$$v_t - \mathcal{M}^-_{\mathcal{L}_0} v \leq m'\varphi - 1 - m\mathcal{M}^-_{\mathcal{L}_0}\varphi - \mathcal{M}^-_{\mathcal{L}_0}(u\chi_{B_1^c}).$$

$\mathcal{M}^-_{\mathcal{L}_0}(u\chi_{B_1^c})(x,t)$ can be easily estimated for $(x,t) \in C_{3/4,1}$, in terms of the sets appearing in the hypothesis of the Lemma by the Tchebyshev inequality, notice that we will be using $\rho \geq 2^k$,

$$\begin{aligned}
\mathcal{M}^-_{\mathcal{L}_0}(u\chi_{B_1^c})(x,t) &\geq \lambda(2-\sigma)\sum_{i=0}^{k-1}\int_{y+x\in B_{2^{i+1}}\setminus B_{2^i}} \frac{u(y+x,t)}{|y|^{n+\sigma}}dy,\\
&\geq C(2-\sigma)M\sum_{i=0}^{k-1} 2^{(2-\sigma)i}\frac{|G_i(t)|}{|B_{2^{i+1}}\setminus B_{2^i}|}
\end{aligned}$$

where $G_i(t) := \{y \in B_{2^{i+1}} \setminus B_{2^i} : u(y,t) > M2^{2i}\}$. So, in order to get that v is a sub solution of the same equation we need,

$$\begin{aligned} m'\varphi - m\mathcal{M}^-_{\mathcal{L}_0}\varphi &\leq F \\ &:= C(2-\sigma)M\sum_{i=0}^{k-1} 2^{(2-\sigma)i}\frac{|G_i(\cdot)|}{|B_{2^{i+1}} \setminus B_{2^i}|} \end{aligned} \tag{4.2.1}$$

This is the moment to fix m. The previous computation suggests us to take m as the solution of the following ordinary differential equation for some $a > 0$ sufficiently large,

$$m' + am = F,$$
$$m(-\Delta t) = 0.$$

Notice that $\mathcal{M}^-_{\mathcal{L}_0}\varphi \geq 0$ if $\varphi \leq \delta$ for some universal $\delta > 0$. In that case the equation automatically implies (4.2.1) as $\varphi \leq 1$. On the other hand, if $\varphi > \delta$, we can also imply (4.2.1) by taking $a = \|(\mathcal{M}^-_{\mathcal{L}_0}\varphi)^-\|_\infty/\delta$.

We finally need to check that we can make $m \geq 2\Delta t$ in $[-\Delta t/2, 0]$ by integrating the previous equation,

$$\begin{aligned} m(t) &= \int_{-\Delta t}^t F(s)e^{-a(t-s)}ds, \\ &= C(2-\sigma)M\sum_{i=0}^{k-1} 2^{(2-\sigma)i}\int_{-\Delta t}^t \frac{|G_i(s)|}{|B_{2^{i+1}} \setminus B_{2^i}|}e^{-a(t-s)}ds. \end{aligned}$$

By the hypothesis of the Lemma we get that, for $t \geq -\Delta t/2$,

$$m(t) \geq C(2-\sigma)\frac{2^{(2-\sigma)k}-1}{2^{2-\sigma}-1}e^{-a\Delta t}\Delta t \geq C2^{(2-\sigma)k}\Delta t,$$

which is larger than $2\Delta t$, independently of $\sigma \in [1,2)$, provided that $(2-\sigma)k$ is sufficiently large. $\square$

Corollary 4.2.2. *Let $k \sim (2-\sigma)^{-1}$, as required for the conclusion in the previous Lemma, $r \in (0,1]$, $\rho \geq 1$, $\Delta t \in (0, (2^{-k}r)^\sigma]$ and let $u \geq 0$ satisfies,*

$$\begin{aligned} u_t - \mathcal{M}^-_{\mathcal{L}_0} u &\geq -C_0 \text{ in viscosity in } C_{2^{-k}r,\Delta t}, \\ u(0,0) &= 0. \end{aligned}$$

Then, given $M > 0$, there exists some non negative integer $i \leq (k-1)$ such that for $r_i = 2^{-i}r$,

$$\frac{|\{u > MC_0 r^{-(2-\sigma)} r_i^2\} \cap (B_{r_i} \setminus B_{r_i/2}) \times (-\Delta t, -\Delta t/2]|}{|(B_{r_i} \setminus B_{r_i/2}) \times (-\Delta t, -\Delta t/2]|} < M^{-1}.$$

The following Corollary follows from the proof of the Lemma 4.2.1 using $k = 1$. Its formulation is closer to the one found in [35].

Corollary 4.2.3 (Weak Point Estimate)**.** *Let $u \geq 0$ satisfies,*

$$\begin{aligned} u_t - \mathcal{M}^-_{\mathcal{L}_0} u &\geq -f \text{ in viscosity in } C_{1,1}, \\ C_0 &:= \int_{-1}^0 \sup_{y \in B_1} f^+(y,s)ds. \end{aligned}$$

Then for every $M > 0$,

$$\frac{|\{u > M\} \cap C_{2,1/2}(0,-1/2)|}{|C_{2,1/2}(0,-1/2)|} < C(2-\sigma)\left(\inf_{C_{1,1/2}} u + C_0\right) M^{-1}$$

4.3 Preliminary Lemmas

Here **we fix some hypothesis and notation** that we will carry for the next results.

1. $\rho \geq 4$ where ρ is the one coming from $\mathcal{K}_0 = \mathcal{K}_0(\sigma, \rho, \lambda, \Lambda)$.

2. $u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f$ in viscosity in $C_{2,1}$ for some f depending only on time and upper semicontinuous.

3. $u \geq 1$ in $\partial_p C_{1,1}$

4. $\inf_{C_{1,1}} u = u(x_0, t_0) \in [-1, 0)$ for some $(x_0, t_0) \in C_{1,1}$.

5. Let Γ be the **parabolic convex envelope of** u supported in B_d for some $d \geq 2$ sufficiently large and to be fixed,

$$\begin{aligned}\Gamma(x,t) :=& \sup\{p \cdot (x - x_0) + h : \\ & \qquad p \cdot (y - x_0) + h \leq -u^-(y,s)\ \forall (y,s) \in C_{d,1+t}(0,t)\}.\end{aligned}$$

6. Let $\partial\Gamma(x,t)$ be the set of **sub differentials** of Γ at (x,t),

$$\partial\Gamma(x,t) := \{p \in \mathbb{R}^n : p \cdot (y - x) + \Gamma(x,t) \leq \Gamma(y,s)\ \forall (y,s) \in C_{d,1+t}(0,t)\}.$$

   Notice that $\partial\Gamma(B_d, t) = \partial\Gamma(B_1, t)$. We denote $|\partial\Gamma(x,t)| := \sup_{p \in \partial\Gamma(x,t)} |p|$.

7. Let $h(\cdot, t) : \partial\Gamma(B_d, t) \to (-\infty, 0]$ be the **Legendre transform of** Γ **centered at** x_0,

$$\begin{aligned}h(p,t) &:= \inf_{y \in B_d} (\Gamma(y,t) - p \cdot (y - x_0)) \\ &= \sup\{h : p \cdot (y - x_0) + h \leq -u^-(y,t)\ \forall y \in B_d\}.\end{aligned}$$

8. Let $\Phi(x,t) := (\partial\Gamma(x,t), h(\partial\Gamma(x,t),t))$.

9. Let $\Sigma := \{u = \Gamma\} \subseteq C_{1,1}$.

10. Given $(p,h) \in \mathbb{R}^n \times \mathbb{R}$, let

$$P_{p,h}(y) = (p \cdot (y - x_0) + h)\chi_{B_2}(y) + \chi_{B_2^c}(y).$$

These are some preliminary lemmas.

Lemma 4.3.1. *Given $p \in \partial\Gamma(B_1,t)$ and $h = h(p,t)$, the following properties hold:*

1. $|p| \leq \frac{1}{(d+1)}$,

2. $-\frac{d+2}{d-1}\theta \leq P_{p,h} \leq 0$ *in* B_2,

3. $\mathcal{M}^-_{\mathcal{K}_0} P_{p,h} \geq 0$ *in* B_1 *provided that* d *is sufficiently large, independently of* $\theta \in (0,1]$ *and* $\sigma \in [1,2)$.

Proof. 1 and 2 follow from the fact that the plane $y \mapsto (p \cdot (y - x_0) + h)$ remains below zero in B_d and crosses the level set $-\theta$ at some point in B_1. Having these two properties we can estimate $\mathcal{M}^-_{\mathcal{K}_0} P_{p,h}$ in B_1 in the following way. For $K \in \mathcal{K}_0$ and $x \in B_1$,

$$\begin{aligned}
L_K P_{p,h}(x) &= \int_{B_1^c} (P_{p,h}(y+x) - P_{p,h}(x))K(y)dy,\\
&\geq \int_{B_2^c(-x)} K(y)dy + \int_{B_2(-x)\setminus B_1} p \cdot (y-x)K(y)dy,\\
&\geq (2-\sigma)\left(C_1 - \frac{C_2}{d-1}\right).
\end{aligned}$$

We have used $\rho \geq 4$ to obtain the lower bound for the integral in $B_2^c(-x) \supseteq B_4 \setminus B_3$. This finally implies 3 by taking d sufficiently large. $\square$

Lemma 4.3.2. *Given* $(t, t+\Delta t] \subseteq (-1, 0]$, *the following properties hold for* h*:*

1. *The domain of* $h(\cdot, t)$ *is non decreasing in time. i.e.*

$$\partial\Gamma(B_1, t) \subseteq \partial\Gamma(B_1, t+\Delta t).$$

2. h *is non increasing in time.*

3. h *is Lipschitz in time. Specifically, for* $p \in \partial\Gamma(x,t)$

$$\begin{aligned} \Delta h &:= h(p, t+\Delta t) - h(p,t), \\ &\geq -C\Delta t(\|f^+\|_{L^\infty((t,t+\Delta t])} + \operatorname{diam} \partial\Gamma(B_1, t)). \end{aligned}$$

For some universal C.

Proof. The first two properties are consequences of the monotonicity of Γ. If at time t, the plane $y \mapsto (p \cdot (y - x_0) + h)$ is a supporting plane for the graph of $\Gamma(\cdot, t)$ then at time $(t + \Delta t)$ it crosses or touches the graph of $\Gamma(\cdot, t+\Delta t) \leq \Gamma(\cdot, t)$ while remaining below zero in B_d. Therefore by lowering h we can find a supporting plane for $\Gamma(\cdot, t+\Delta t)$ with the same slope p.

For the last part we fix $p \in \partial\Gamma(B_1, t)$ and $h = h(p,t)$. Assume that $\Delta h < 0$ and consider the following test function,

$$v(y,s) = P_{p,h}(y) + \frac{\Delta h}{2\Delta t}(s-t)$$

By this definition, v has to cross Γ in $B_2 \times \{t+\Delta t\}$. By the definition of Γ, $\left(P_{p,h} + \frac{\Delta h}{2\Delta t}\right)$ also has to cross u in $C_{1,\Delta t}(0, t+\Delta t)$ meanwhile remaining below u in $\partial_p C_{1,\Delta t}(0, t+\Delta t)$. Let $t_1 \in (t, t+\Delta t]$ the last time when $\left(P_{p,h} + \frac{\Delta h}{2\Delta t}\right) < u$,

$$t_1 = \sup\left\{s \in (t, t+\Delta t] : \left(P_{p,h} + \frac{\Delta h}{2\Delta t}\right)(\cdot, s) < u(\cdot, s)\right\}.$$

Then $\tilde{v}(y,s) = v(y, s-(t+\Delta t - t_1))$ is a test function touching u from below at some $(x_1, t_1) \in C_{1,\Delta t}(0,t)$. Plugging it into the equation for u and using Lemma 4.3.1 we obtain that,

$$-f(t_1) \leq \tilde{v}_t(x_1, t_1) - \mathcal{M}^-_{\mathcal{L}_0}\tilde{v}(x_1,t_1) \leq \frac{\Delta h}{2\Delta t} + \beta|p|,$$

which concludes the proof. $\square$

Corollary 4.3.3. *Given $p \in \partial\Gamma(B_1, t)$ and $h = h(p,t)$, then*

$$P_{p,h} - C\Delta t(\|f^+\|_{L^\infty((t,t+\Delta t])} + \operatorname{diam}\partial\Gamma(B_1,t)) \leq \Gamma \text{ in } C_{1,1+t+\Delta t}(0, t+\Delta t).$$

Lemma 4.3.4. *Let $\Gamma : C_{r,\Delta t} \to \mathbb{R}$ be a parabolic convex function such that*

$$|\{\Gamma \geq M\} \cap (B_r \setminus B_{r/2}) \times (-\Delta t, -\Delta t/2]| \leq \varepsilon_0 |(B_r \setminus B_{r/2}) \times (-\Delta t, 0]|.$$

Then $\Gamma \leq M$ in $C_{r/2,\Delta t}(0, \Delta t/2)$ provided that ε_0 is sufficiently small, depending only on the dimension.

Proof. By the convexity and monotonicity of Γ we can assume that

$$N := \sup_{C_{r/2,\Delta t/2}} \Gamma = \Gamma(r/2e_1, -\Delta t/2) \leq \Gamma \text{ in } A,$$

where

$$A = \{(x,t) \in (B_r \setminus B_{r/2}) \times (-\Delta t, -\Delta t/2] : x \cdot e_1 > r/2\}.$$

Then we just need to have ε_0 smaller than $|A|/|(B_r \setminus B_{r/2}) \times [-\Delta t, -\Delta t/2]| \sim 1$, to obtain that the hypothesis of the Lemma implies $N \leq M$. □

4.4 Covering the Contact Set

We show in the next two Lemmas how to cover the contact set Σ with pieces where u detaches from Γ in a controlled way. For this we use the Key Lemma 4.2.1 and the tools from the previous section. The first result finds a configuration for each point in Σ meanwhile the second Lemma provides an algorithm which produces a covering with some desired properties.

Lemma 4.4.1. *Let $k \sim (2-\sigma)^{-1}$ as in Lemma 4.2.1, $r \in (0,1]$, $\Delta t \in (0, (2^{-(k+1)}r)^2]$, $(x,t) \in \Sigma$, $p \in \partial\Gamma(x,t)$, $h = h(p,t)$. There exists some non negative integer $i \leq (k-1)$, such that the following holds for $r_i = 2^{-i}r$ and,*

$$G := \|f^+\|_{L^\infty(C_{r_k,\Delta t}(x,t))} + \operatorname{diam} \partial\Gamma(B_1, t),$$
$$B := \|f^+\|_{L^\infty(C_{1,\Delta t}(0,t+\Delta t/2))} + \operatorname{diam} \partial\Gamma(B_1, t).$$

1. ***Control for u detaching from $P_{p,h}$:*** *For some universal C and ε_0 as in the Lemma 4.3.4,*

$$\frac{|\{u \geq P_{p,h} + CC_0 r^{-(2-\sigma)} r_i^2\} \cap (B_{r_i}(x) \setminus B_{r_i/2}(x)) \times (t-\Delta t, t-\Delta t/2]|}{|(B_{r_i}(x) \setminus B_{r_i/2}(x)) \times (t-\Delta t, t-\Delta t/2]|} \leq \varepsilon_0.$$

2. ***Flatness for*** Γ***:*** *In* $C_{r_i/2,\Delta t}(x, t + \Delta t/2)$

$$-CC_0 r_i^2 < \Gamma - P_{p,h} \leq CC_0 r^{-(2-\sigma)} r_i^2.$$

3. ***Control of the jacobian measure of*** Φ***:***

$$|\Phi(C_{r/4,\Delta t}(x, t + \Delta t/2))| \leq CC_0^{n+1} r^{-(2-\sigma)n} |C_{r/4,\Delta t}(x, t + \Delta t/2)|.$$

Remark 4.4.2. *In the next proof we are using that the equation for* u *holds in* $C_{1,(3/2)\Delta t}(x, t + \Delta t/2)$ *but we do not necessarily know that* $C_{1,(3/2)\Delta t}(x, t + \Delta t/2) \subseteq C_{2,1}$*. This is from where it comes the decision we made to impose the equation in* $C_{2,1}$ *instead of* $C_{1,1}$*. The inclusion in time is a technical issue. The correct statement of the previous Lemma should replace* $t - \Delta t$*,* $t - \Delta t/2$ *by* -1 *whenever they go before* -1 *and should also replace* $t + \Delta t/2$ *by 0 whenever it goes after 0. Notice that the constants remain uniform when* $\Delta t \to 0$*. As we said, this is a technical detail that we decided to omit in favor of a lighter notation.*

Proof. To prove 1 we apply the Corollary 4.2.2 to $(u - P_{p,h})$ in $C_{2^{-(k+1)}r,\Delta t}(x, t)$. By Theorem 3.2.3 and Lemma 4.3.1 we know that u satisfies in viscosity in $C_{1,1}(x, 0) \supseteq C_{2^{-(k+1)}r,\Delta t}(x, t)$,

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f + \mathcal{M}^-_{\mathcal{K}_0} P_{p,h} - \beta|p| \geq -CC_0.$$

This proves 1.

To prove 2 we notice first that the previous estimate for $(u - P_{p,h})$ also holds for $(\Gamma - P_{p,h}) \leq (u - P_{p,h})$. Then by Lemma 4.3.4 we get the upper bound $\Gamma - P_{p,h} \leq CC_0 r^{-(2-\sigma)} r_i^2$ in $C_{r_i/2,\Delta t}(x, t+\Delta t/2)$. The lower bound holds because of Corollary 4.3.3 and the fact that $\Delta t \leq (2^{-(k+1)}r)^2 < r_i^2$.

As a consequence of the bounds given by 2 and the geometry of convex functions we get that $\operatorname{diam}(\partial\Gamma(B_{r_i/4}(x) \times \{t+\Delta t/2\})) \leq CC_0 r^{-(2-\sigma)} r_i$. Then by Lemma 4.3.2,

$$\begin{aligned}
\Phi(C_{r_i/4,\Delta t}(x, t+\Delta t/2)) \subseteq Cylinder&,\\
&:= \{(p', h') : p' \in \partial\Gamma(B_{r_i/4}(x) \times \{t+\Delta t/2\}),\\
&h' \in [h(p', t), h(p', t) + CC_0 \Delta t]\}.
\end{aligned}$$

For which it is easy to verify that $|Cylinder| \leq CC_0^{n+1} r^{-(2-\sigma)n} r_i^n \Delta t$. This concludes 3 and the Lemma. □

Lemma 4.4.3 (Covering Lemma for the Contact Set)**.** *Let $k \sim (2-\sigma)^{-1}$ as in Lemma 4.2.1, $r \in (0,1]$ and $\Delta t \in (0, (2^{-(k+1)}r)^2]$. There exists a finite covering of Σ by non overlapping boxes $\{K_j\}$ such that:*

1. *$K_j := Q_j \times I_j$ with $Q_j \subseteq \mathbb{R}^n$ an open cube with diameter $d_j \leq r/4$ and $I_j = (-l_j \Delta t/2, -(l_j - 1)\Delta t/2]$ with $l_j \in \mathbb{N}$,*
2. *$\bar{K}_j \cap \Sigma \neq \emptyset$,*
3. *$|\{u < \Gamma + Cr^{-(2-\sigma)} F(\tilde{I}_j) d_j^2\} \cap \tilde{K}_j| > \mu |\tilde{K}_j|$, where*

$$\begin{aligned}
\tilde{K}_j &:= 16\sqrt{n} Q_j \times \tilde{I}_j,\\
\tilde{I}_j &:= (-(l_j+2)\Delta t/2, -(l_j - 1)\Delta t/2],\\
F(\tilde{I}_j) &:= \|f^+\|_{L^\infty(\tilde{I}_j)} + \operatorname{diam} \partial\Gamma(B_1, -(l_j - 1)\Delta t/2)
\end{aligned}$$

4. $|\Phi(K_j)| \leq Cr^{-(2-\sigma)n} F(\tilde{I}_j)^{n+1} |K_j|$,

For some universal constants $C > 0$ and $\mu \in (0,1)$ independent of $\sigma \in [1,2)$.

Proof. Fix a time interval I_l and consider a covering of $B_1 \times I_l$ by rectangles $\{K = Q \times I_l\}$ with $\mathrm{diam}(Q) = r/4$. Discard every rectangle K such that $\bar{K} \cap \Sigma = \emptyset$. Whenever $Q \times I_l$ does not satisfy 4 or 3, we split Q into 2^n congruent cubes $\{Q'\}$ and consider now the rectangles given by $\{K' = Q' \times I_l\}$. We need to prove that eventually all rectangles produced by this algorithm satisfy 3 and 4 and therefore the process finishes after a finite number of steps. In fact we will show that it will finish before $k \sim (2-\sigma)^{-1}$ iterations.

Let $\bar{Q}_1 \times I_l \supseteq \bar{Q}_2 \times I_l \supseteq \ldots \supseteq \bar{Q}_k \times I_l \ni (x,t)$ such that $(x,t) \in \Sigma$. Let also $p \in \partial\Gamma(x,t)$ and $h = h(p,t)$. From the Lemma 4.4.1 there exists some non negative integer $i \leq k$, such that for $r_i = 2^{-i} r$,

$$\frac{|\{u \geq P_{p,h} + CF(\tilde{I}_j) r^{-(2-\sigma)} r_i^2\} \cap (B_{r_i}(x) \setminus B_{r_i/2}(x)) \times (t-\Delta t, t-\Delta t/2]|}{|(B_{r_i}(x) \setminus B_{r_i/2}(x)) \times (t-\Delta t, t-\Delta t/2]|} \leq \varepsilon_0, \tag{4.4.2}$$

and

$$\Phi(C_{r_i/4,\Delta t}(x, t+\Delta t/2)) \leq Cr^{-(2-\sigma)n} F(\tilde{I}_j)^{n+1}. \tag{4.4.3}$$

One of the previous rectangles, $K_j = Q_j \times I_l$, satisfies $r_i/8 < \mathrm{diam}(Q_j) \leq r_i/4$. This implies that $Q_j \times I_l \subseteq C_{r_i/4,\Delta t}(x, t+\Delta t/2)$ and $(B_{r_i}(x) \setminus B_{r_i/2}(x)) \times (t-\Delta t, t-\Delta t/2] \subseteq \tilde{K}_j$.

3 and 4 now follow from (4.4.2), (4.4.3), the previous inclusions and the fact that we can replace $P_{p,h}$ by $\Gamma \geq P_{p,h}$ in (4.4.2). $\square$

4.5 Alexandrov-Bakelman-Pucci-Krylov-Tso Type of Estimates

Next we give two type of estimates, both controlling $\sup_{C_{1,1}} u^-$. The first one does it by the L^{n+1} norm of a right hand side which only depends in time. The second one requires intend a well localized right hand side and concludes that the measure where u is not too large is not trivial. This second result is the one we will need in the next chapter to prove our Point Estimate.

Theorem 4.5.1. *Let $\rho \geq 4$ (coming from $\mathcal{K}_0$), $\sigma \in [1,2)$, $f \in USC((-1,0])$ and u satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f \text{ in viscosity in } C_{2,1},$$

$$u \geq 0 \quad \text{in } \partial_p C_{1,1}.$$

Then, for some universal constant $C > 0$, independent of $\sigma \in [1,2)$,

$$\sup_{C_{1,1}} u^- \leq C \|f^+\|_{L^{n+1}((-1,0])}.$$

Lemma 4.5.2. *Let $\rho \geq 4$, $\sigma \in [1,2)$, $a \in (0,1]$, $f \in USC((-1,0])$ and u satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f \text{ in viscosity in } C_{2,1},$$

$$u \geq 1 \text{ in } C_{2,1} \setminus C_{1,a},$$

$$\sup_{C_{1,1}} u^- = -u(x_0,t_0) = \theta \in (0,1] \text{ for some } (x_0,t_0) \in C_{1,1}$$

Then, for some universal constant $C > 0$ independent of $\sigma \in [1,2)$, we have that

$$\theta^{n+1} \leq C \int_a^0 (f^+(t) + \operatorname{diam} \partial\Gamma(B_1,t))^{n+1} dt.$$

Where Γ is parabolic convex envelope of u, supported in a sufficiently large ball B_d (see Lemma 4.3.1).

Proof. We have as in [37] that,

$$\Phi(\{\Gamma = u\}) \supseteq Cone := \{(p,h) : h \in (-\theta, 0), |p| < |h|/(d+1)\}.$$

We use the disjoint covering $\{K_j\}$ from Lemma 4.4.3 with $r = 1$,

$$\theta^{n+1} \leq C|\Phi(\cup_j K_j)| \leq C \sum_j |\Phi(K_j)|,$$

We group now the previous sum in each interval $J_l = (-l\Delta t/2, -(l-1)\Delta t/2)$,

$$\begin{aligned}
\theta^{n+1} &\leq C \sum_l \left(F(\tilde{J}_l)^{n+1} \sum_{I_j = J_l} |K_j| \right), \\
&= C \sum_l \left(F(\tilde{J}_l)^{n+1} \left| \bigcup_{I_j = J_l} K_j \right| \right), \\
&\leq C \sum_l F(\tilde{J}_l)^{n+1}.
\end{aligned}$$

Because $\cup_{I_j = J_l} K_j \subseteq B_{5/4}$. Sending $\Delta t \to 0$ we obtain the desired estimate. Notice that $\operatorname{diam} \partial\Gamma(B_1, \cdot)$ is a bounded upper semicontinuous function too, so that the Riemann sum above converges to the desired integral. $\square$

Proof of Theorem 4.5.1. Assume without loss of generality that $\sup_{C_{1,1}} u^- \geq 2$. Let, for $\theta \in (0,1]$,

$$\begin{aligned}
u_\theta &= u + \sup_{C_{1,1}} u^- - \theta, \\
a_\theta &= \sup\{t \in [-1, 0] : \inf_{x \in B_1} u_\theta(x,t) \leq 0\}.
\end{aligned}$$

By the previous Lemma,

$$\theta^{n+1} \leq C \int_{a_\theta}^{0} (f^+(t) + \operatorname{diam} \partial\Gamma_\theta(B_1,t))^{n+1} dt.$$

Now we take $\theta(i)^{n+1} = 1/i$ and add the inequalities for $i = 1, 2, \ldots, N$, ($a_i = a_{\theta(i)}, a_{N+1} = 0, \Gamma_i = \Gamma_{\theta(i)}$),

$$\begin{aligned}
\ln N &\leq C \sum_{i=1}^{N} \int_{a_i}^{0} (f^+(t) + \operatorname{diam} \partial\Gamma_i(B_1,t))^{n+1} dt, \\
&\leq C \left(N \|f^+\|_{L^{n+1}((-1,0])}^{n+1} + \sum_{i=1}^{N} \sum_{j=1}^{i} \int_{a_i}^{a_{i+1}} (\operatorname{diam} \partial\Gamma_j(B_1,t))^{n+1} dt \right).
\end{aligned}$$

Now we notice that, by the previous definitions, $\sup \Gamma_j^- = \theta(i)$ in $[-1, a_i]$ for $j \leq i$. This implies, by the same arguments as in the proof of Lemma 4.3.1, that $(\operatorname{diam} \partial\Gamma_j(B_1,t))^{n+1} \leq C(\theta(i))^{n+1} = C/i$. Then the double sum can be replaced by a universal constant and we just need to fix N sufficiently large so that the left hand side absorbs this constant. This concludes the proof. $\square$

Theorem 4.5.3. *Let $\rho \geq 4, \sigma \in [1,2)$ and u satisfies,*

$$\begin{aligned}
u_t - \mathcal{M}^-_{\mathfrak{L}_0} u &\geq -C_0 \chi_{B_{1/8}} && \text{in viscosity in } C_{2,1}, \\
u &\geq 0 && \text{in } \partial_p C_{1,1}.
\end{aligned}$$

Then, for some universal constants $C, M > 0$, independent of $\sigma \in [1,2)$,

$$\sup_{C_{1,1}} u^- \leq C \left(\sup_{C_{1,1}} u^- + C_0 \right) \left| \left\{ u < \left(\sup_{C_{1,1}} u^- + C_0 \right) M \right\} \cap C_{1/2,1} \right|^{1/(n+1)}.$$

Lemma 4.5.4. *Let $\rho \geq 4, \sigma \in [1,2), C_0 > 0$ and u satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq 0 \text{ in viscosity in } C_{1,1} \setminus C_{1/8,1},$$

$$u \geq 0 \text{ in } \partial_p C_{1,1}$$

$$\inf_{C_{1,1}} u \geq -2.$$

Then, there is some $\alpha > 2$ sufficiently large and independent of σ such that for all $(y,s) \in \mathbb{R}^n \times [-1,0]$,

$$\varphi(y) = \left(\left(\frac{(|y| - 1/8)^+}{2}\right)^\alpha - 2\right)\chi_{B_2}(y) \leq u(y,s).$$

Proof. It suffices to show that $\mathcal{M}^-_{\mathcal{L}_0}\varphi \geq 0$ in $B_1 \setminus \bar{B}_{1/2}$, for α sufficiently large, in order to get $\varphi \leq u$ by the comparison principle.

Let $x \in B_1 \setminus \bar{B}_{1/8}$ with $r = |x|$. From the computations proving the Proposition 2.1.2, we get,

$$\begin{aligned}
\mathcal{M}^-_{\mathcal{L}_0}\varphi(x) &\geq \frac{\alpha(r-1/8)^{\alpha-2}}{2^\alpha}\left(\lambda\left((\alpha-1) + (n-1)\frac{r-1/8}{r}\right) - \beta(r-1/8)\right) \\
&\quad - C(2-\sigma), \\
&\geq \frac{\alpha(r-1/2)^{\alpha-2}}{2^\alpha}\left(\lambda(\alpha-1) - \beta\right) - C(2-\sigma),
\end{aligned}$$

for some constant $C > 0$, independent of x and α. Then we make α sufficiently large and $(2-\sigma)$ sufficiently small to get the desired inequality.

Now we look at the case where $\sigma \in [1, 2-\varepsilon)$ is away from two. φ is convex in B_2 and $\delta\varphi(x;y) \geq 0$ for $x, y \in B_1$. Therefore we just need to consider the following integrals outside B_1 for $L_{K,\beta} \in \mathcal{L}_0$,

$$\begin{aligned}
L_{K,\beta}\varphi(x) &\geq \int_{B_1^c} (\varphi(y+x) - \varphi(x))K(y)dy - \beta|D\varphi(x)|,\\
&\geq \int_{B_1^c \cap B_2(-x)} (\varphi(y+x) - \varphi(x))K(y)dy\\
&- \varphi(x) \int_{B_2^c(-x)} K(y)dy - \beta|D\varphi(x)|,\\
&\geq -C_1 \sup_{B_1} \varphi - C_2 \operatorname*{osc}_{B_2} \varphi - \beta\|D\varphi\|_{L^\infty(B_1)}.
\end{aligned}$$

For some constants $C_1, C_2 > 0$, independent of K and $\sigma \in [1, 2-\varepsilon)$. As α becomes larger, $\operatorname{osc}_{B_2} \varphi$ and $\|D\varphi\|_{L^\infty(B_1)}$ go to zero meanwhile $-\sup_{B_1} \varphi$ goes to two. This concludes the Lemma. □

Proof of Theorem 4.5.3. Let φ as in the previous Lemma and consider $U = (2u/(\sup_{C_{1,1}} u^-) - \varphi(1/4))$, so that U satisfies the hypothesis of the previous section with $\inf_{C_{1,1}} U = -1/16^\alpha$ universal. As we did in the proof of Lemma 4.5.2 we apply Lemma 4.4.3 but now with $r = 1/(64\sqrt{n})$ to have that $\tilde{K}_j \subseteq C_{1/2,1}$,

$$1 \leq C \sum_l \left(F(\tilde{J}_l)^{n+1} \left| \bigcup_{I_j = I_l} K_j \right| \right).$$

By Besicovitch, we can extract a subset from $\{\tilde{K}_j\}_{I_j=I_l}$ (denoted by the same) which covers $\cup_{I_j=I_l} K_j$ and has the finite intersection property,

$$\begin{aligned}
\left|\bigcup_{I_j=I_l} K_j\right| &\leq \sum_{I_j=I_l} |\tilde{K}_j|, \\
&< C \sum_{I_j=I_l} |\{U < F(\tilde{J}_l)C\} \cap \tilde{K}_j|, \\
&\leq C|\{U < F(\tilde{J}_l)C\} \cap B_{1/2} \times \tilde{I}_l|, \\
&\leq C|\{U < F(\tilde{J}_l)C\} \cap C_{1/2,1}|.
\end{aligned}$$

Where we have used the third property stated in Lemma 4.4.3.

Finally we just have to see that,

$$\begin{aligned}
F(\tilde{J}_l) &= \left(\frac{2C_0}{\sup_{C_{1,1}} u^-} + \operatorname{diam} \partial\Gamma(B_1, \tilde{I}_l) \right), \\
&\leq C \left(\frac{C_0}{\sup_{C_{1,1}} u^-} + 1 \right),
\end{aligned}$$

because $\inf_{C_{1,1}} \Gamma = \inf_{C_{1,1}} U$, which, as we already said, is universal. $\square$

Chapter 5

Krylov-Safanov Estimates

The goal of this chapter is to prove the Hölder estimates established in Theorems 5.3.1, 5.4.2 and 5.4.3. The main tool required is a Point Estimate or L^{ε} Lemma which would play the role that the Mean Value formula did in Chapter 1 for the heat equation.

As in the elliptic, second order case, the proof uses the Calderón-Zygmund decomposition of the level sets in order to prove that they decrease in a controlled way as the level set increases, being the first step a consequence of the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate (Theorem 4.5.3) of the previous chapter, see [17]. There are however two main difficulties that arise, one coming from the time dependence and the other coming from scaling of the equation.

The estimates of the previous chapter control the size of the set where the super solution is negative in $C_{1,1}$ but they say nothing about what happens in times after $t = 0$. This will bring a shift in time which have to be controlled by the iteration.

On the other hand, our equation scales with order $\sigma \in [1, 2)$. For classical parabolic equations the appropriated Calderon-Zygmund decomposition is done by splitting the space cubes in halves meanwhile it splits the time intervals in fourths. This preserves the scaling ratio of parabolic equations of order two. For $\sigma \in (1, 2)$ there is not an obvious way to do such a splitting. Our way to handle it is by modifying the splitting algorithm such that it splits the time interval by halves or fourths according to the scaling ratio of the original box. By this reason we need to introduce an extra parameter $\tau \in [1, 4]$ which keeps track of this scaling ratio at every possible step.

5.1 Point Estimate

This is the Uniform Point Estimate we will proof in next few sections.

Theorem 5.1.1 (Uniform Point Estimate)**.** *Let $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f \text{ in viscosity in } C_{2,2}(0,1).$$

Then for every $\alpha > 0$,

$$\frac{|\{u \geq \alpha\} \cap C_{1,1}|}{|C_{1,1}|} \leq C \left(\inf_{C_{1,1}(0,1)} u + \|f^+\|_\infty \right)^\varepsilon \alpha^{-\varepsilon},$$

for some contents $C, \varepsilon > 0$ independent of $\sigma \in [1, 2)$.

5.1.1 Initial Configurations

Lemma 5.1.2 (Special Function)**.** *There exits a smooth function φ such that,*

$$\begin{aligned} \varphi_t - \mathcal{M}^-_{\mathcal{L}_0} \varphi &\leq -1 \text{ in } C_{2\sqrt{n},37}(0,36) \setminus C_{1/8,1}, \\ \varphi &\leq 0 \quad \text{in } \partial_p C_{2\sqrt{n},37}(0,36), \\ \varphi &\geq 2 \quad \text{in } K_{3,36}(0,36). \end{aligned}$$

Provided that σ is sufficiently close to two.

Proof. Loosely speaking, the Lemma says that φ should behave similarly to the *fundamental solution* in the sense that it allows φ to grow a fix amount at $K_{3,36}(0,36)$ by concentrating the positive mass of its right hand side in the intermediate region $C_{1/8,1}$. This is our initial ansatz,

$$\varphi_1(y,s) := (s+1)^{-\alpha^3}\Phi(y(s+1)^{-1/2}),$$
$$\Phi(z) := \exp(-(\alpha/2)|z|^2).$$

Step 1: $(\varphi_1)_t + \beta|D\varphi_1| - \inf_{A\in[\lambda',\Lambda']}\operatorname{tr}(AD^2\varphi_1) \leq -\alpha^{3/2}(s+1)^{-(\alpha^3+2)}\Phi$ in $C_{2\sqrt{n},37}(0,36)\setminus C_{1/8,1}$ provided that α is sufficiently large.

$$(\varphi_1)_t = \alpha(s+1)^{-(\alpha^3+1)}(-\alpha^2+(1/2)|z|^2)\Phi,$$
$$|D\varphi_1| = \alpha(s+1)^{-(\alpha^3+1/2)}|z|\Phi,$$
$$D^2\varphi_1 = \alpha(s+1)^{-(\alpha^3+1)}(z\otimes z - Id)\Phi.$$

For $(y,s)\in C_{1,1}\setminus C_{1/8,1}$ we have that $|z|\geq 1/8$, then for α sufficiently large,

$$\inf_{A\in[\lambda',\Lambda']}\operatorname{tr}(AD^2\varphi_1) = \alpha(s+1)^{-(\alpha^3+1)}(\lambda'(\alpha|z|^2-1)-\Lambda'(n-1))\Phi,$$
$$\geq \frac{\lambda'\alpha^2}{2}(s+1)^{-(\alpha^3+1)}|z|^2\Phi.$$

Then we see that this term controls all the other terms in $C_{1,1}\setminus C_{1/8,1}$.

In $C_{2\sqrt{n},36}(0,36)$ we can not use anymore that y is away from zero. We use instead the "good" term coming from $(\varphi_1)_t$ and the fact that $(s+1)\sim 1$

and $|z| \leq 2\sqrt{n}$. Again, it is not difficult to see that the leading order will be $-\alpha^3(s+1)^{-(\alpha^3+1)}$ provided that α is sufficiently large.

Step 2: Let ψ a smooth function such that $\psi = 0$ in $\partial_p C_{2\sqrt{n},37}(0,36) \cup (\mathbb{R}^n \times [-1,-1/2])$ and $\psi = 1$ in $K_{3,36}(0,36)$. We consider now $\varphi_2 := \psi\varphi_1$, by the previous step,

$$\begin{aligned}(\varphi_2)_t + \beta|D\varphi_2| - \inf_{A\in[\lambda',\Lambda']} \operatorname{tr}(AD^2\varphi_2) &\leq \psi(\varphi_t + \beta|D\varphi| - \inf_{A\in[\lambda',\Lambda']} \operatorname{tr}(AD^2\varphi)) \\ &+ \varphi(\psi_t + \beta|D\psi| - \inf_{A\in[\lambda',\Lambda']} \operatorname{tr}(AD^2\psi)) \\ &+ 2 \inf_{A\in[\lambda',\Lambda']} \operatorname{tr}(AD\psi D\varphi), \\ &\leq C(-\alpha^{3/2}(s+1)^{-2} + 1 + \alpha(s+1)^{-1}) \\ &(s+1)^{-\alpha^3}\Phi\end{aligned}$$

Once again, α sufficiently large guarantees that the right hand side above is non positive.

Step 3: Let

$$\varphi(y,s) := C\left(\varphi_2(y,s) - (1/2)\left(\inf_{K_{3,36}(0,36)} \varphi_2\right)(s+1)\right).$$

C is chosen sufficiently large so that

$$\begin{aligned}\varphi_t + \beta|D\varphi| - \inf_{A\in[\lambda',\Lambda']} \operatorname{tr}(AD^2\varphi) &\leq -2 \text{ in } C_{2\sqrt{n},37}(0,36) \setminus C_{1/8,1}, \\ \varphi &\geq 2 \quad \text{in } K_{3,36}(0,36).\end{aligned}$$

Notice now that in the closure of $C_{2\sqrt{n},37}(0,36) \setminus C_{1/8,1}$, as σ goes to two, $\varphi_t - \mathcal{M}^-_{\mathcal{L}_0}\varphi$ goes uniformly below -1. This is how close we need to have σ to conclude the Lemma. $\square$

Lemma 5.1.3 (Base configuration)**.** *Let σ as in Lemma 5.1.2 and $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \text{ in viscosity in } C_{2\sqrt{n},37}(0,36),$$
$$\inf_{K_{3,36}(0,36)} u \leq 1.$$

Then

$$|\{u \geq M_1\} \cap K_{1,1}| \leq \mu_1 |K_{1,1}|,$$

for some universal constants $\mu_1 \in (0,1)$, $M_1 > 0$ independent of σ.

Proof. Let φ as in the previous Lemma. The difference $v = (u - \varphi)$ satisfies,

$$v_t - \mathcal{M}^-_{\mathcal{L}_0} v \geq -C\chi_{C_{1/8,1}} \text{ in viscosity in } C_{2\sqrt{n},37}(0,36),$$
$$\inf_{C_{2\sqrt{n},37}(0,36)} v = v(x,t) \in [-C,-1] \quad \text{for some } (x,t) \in C_{1/8,1}.$$

We apply then Theorem 4.5.3 to conclude the Lemma. □

Corollary 5.1.4 (With a parameter τ)**.** *Let σ as in Lemma 5.1.2, $\tau \in [1,4]$ and $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \text{ in viscosity in } C_{2\sqrt{n},(3^\sigma+1)\tau}(0,3^\sigma\tau),$$
$$\inf_{K_{3,3^\sigma\tau}(0,3^\sigma\tau)} u \leq 1.$$

Then, for $\mu_2 = \frac{3+\mu_1}{4}$ and $M_2 = M_1$ from the previous Lemma, we have that,

$$|\{u \geq M_2\} \cap K_{1,\tau}| \leq \mu_2 |K_{1,\tau}|.$$

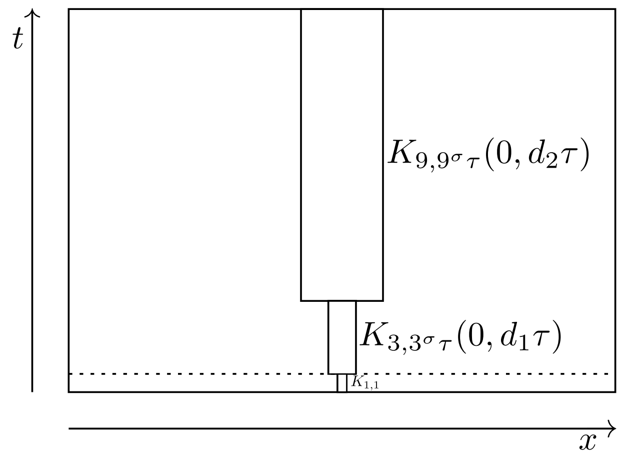


Figure 5.1: Geometric configuration of Corollary 5.1.5

Corollary 5.1.5 (Iteration)**.** *Let σ as in Lemma 5.1.2, $\tau \in [1,4]$, $m \in \mathbb{N}$, $d_i = \sum_{j=1}^{i} 3^{\sigma j}$, and $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \text{ in } C_{3^{\sigma(m-1)}(2\sqrt{n}),d_m+\tau}(0, d_m\tau),$$

$$\inf_{\cup_{i=1}^m K_{3^i,3^{\sigma i}\tau}(0,d_i\tau)} u \leq 1.$$

Then, for $\mu_3 = \mu_2$ and $M_3 = 9M_2$ from the previous Corollary, we have that,

$$|\{u \geq M_3^m\} \cap K_{1,\tau}| \leq \mu_3 |K_{1,\tau}|.$$

5.1.2 A Covering Lemma from the Calderón - Zygmund Decomposition

We are ready to describe the modified Calderón-Zygmund decomposition suited for our problem of order $\sigma \in [1,2)$.

We start with the box $K_{1,1}$. In each step we consider one of the boxes

$K_{r,r^\sigma\tau}(x,t)$ we have already produced and divide $Q_r(x)$ in 2^n congruent cubes in space. With respect to the time interval we do the following:

1. If $1 \leq \tau < 2$ then we subdivide $[t - r^\sigma\tau, t]$ in 2 congruent intervals.

2. If $2 \leq \tau \leq 4$ then we subdivide $[t - r^\sigma\tau, t]$ in 4 congruent intervals.

Finally we take the cartesian product to form the new generation of dyadic boxes from $K_{r,r^\sigma\tau}(x,t)$.

This procedure verifies that if $\tau \in [1,4]$, then the boxes that $K_{r,r^\sigma\tau}(x,t)$ generates have side length $r/2$ (in space) and $(r/2)^\sigma\tau'$ (in time) for some $\tau' \in [1,4]$.

Given two dyadic boxes K and $\tilde{K}$ we say that $\tilde{K}$ is the predecessor of K if K is one of the boxes obtained from the decomposition of $\tilde{K}$.

Lemma 5.1.6 (Dyadic covering)**.** *Let $A \subseteq K_{1,1}$ and $\mu \in (0,1)$, such that*

$$|A| \leq \mu |K_{1,1}|.$$

Then there exists a set of disjoint dyadic boxes $\{K_j\}$ such that:

1. $|A \setminus \cup_j K_j| = 0$,

2. $|A \cap K_j| > \mu |K_j|$,

3. $|A \cap \tilde{K}_j| \leq \mu |\tilde{K}_j|$.

Proof. Starting with $K_{1,1}$, we consider the dyadic boxes, obtained with the previous algorithm, that intersect A but capture a fraction of A smaller than or equal to μ. During this process we select those boxes $\{K_j\}$ that capture a fraction bigger than μ. See figure 5.2.

At the initial step we know that $K_{1,1}$ captures a fraction of A smaller than or equal to μ, therefore we know that $K_{1,1}$ is subdivided and that for any predecessor $\tilde{K}_j \subseteq K_{1,1}$.

This process selects a family of disjoint boxes $\{K_j\}$ that satisfy 2 and 3. To verify 1 we use the Lebesgue Differentiation Theorem. For each $(x,t) \in A \setminus \cup_j K_j$ there exist a family of dyadic boxes $\{K_i^{(x,t)} = K_{r_i, r_i^\sigma \tau_i}(x_i, t_i)\}_{i \geq 1}$ such that,

1. $(x,t) \in A \cap K_i^{(x,t)}$,

2. $r_i \to 0$ as $i \to \infty$ and $\tau_i \in [1,4]$,

3. $|A \cap K_i^{(x,t)}| \leq \mu_1 |K_i^{(x,t)}|$.

From $K_i^{(x,t)} = K_{r_i, r_i^\sigma \tau_i}(x_i, t_i)$ we construct a box with exactly the scale σ, $\bar{K}_i^{(x,t)} = K_{\rho_i, \rho_i^\sigma}(x_i, t_i) \supseteq K_i^{(x,t)}$ such that $\rho_i = r_i \tau_i^{1/\sigma}$. They satisfy instead,

1. $(x,t) \in \bar{K}_i^{(x,t)}$,

2. $\rho_i \to 0$ as $i \to \infty$,

3. $|A \cap \bar{K}_i^{(x,t)}| \leq \bar{\mu} |\bar{K}_i^{(x,t)}|$ with $\bar{\mu} = \frac{3+\mu}{4} < 1$.

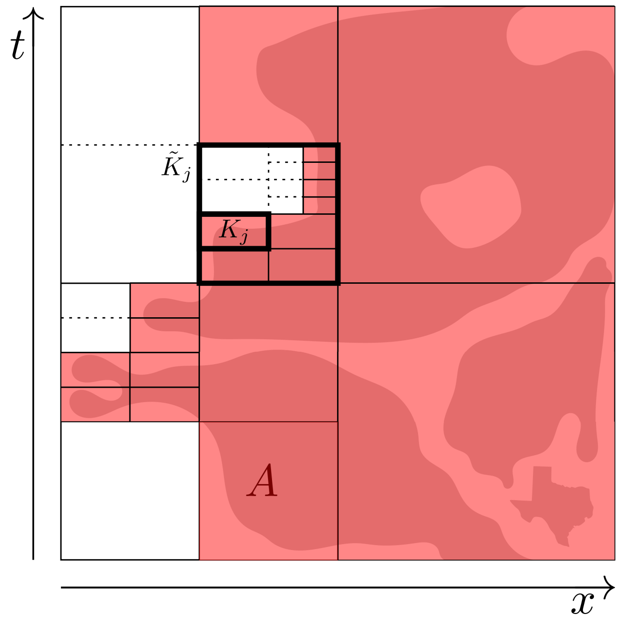


Figure 5.2: Dyadic covering of a set A

These are enough hypothesis to apply a modified version of the Lebesgue Differentiation Theorem and conclude that $|A \setminus \cup_j K_j| = 0$. See for instance Exercise 3 in Chapter 7 of [41]. $\square$

Because the geometric configuration of Lemma 5.1.3 we need to introduce a new tool which handles a shift in time. Given a box $K = Q \times (t-r, t]$ and a natural number $m \geq 1$ we define the **stack** $K^m := Q \times (t, t+mr]$.

Lemma 5.1.7. *Let $A \subseteq K_{1,1}$ and $\mu \in (0,1)$, such that*

$$|A| \leq \mu |K_{1,1}|.$$

Then there exists a set of disjoint dyadic boxes $\{K_j\}$ such that:

1. $\left|\bigcup_j K_j \setminus A\right| = 0$,

2. $|A \cap K_j| > \mu |K_j|$,

3. $|A| \leq \dfrac{(m+1)\mu}{m} \left|\bigcup_j (\tilde{K}_j)^m\right|$.

Proof. Select the same covering $\{K_j\}$ of A from Lemma 5.1.6. Consider a disjoint sub covering of $\{\tilde{K}_j\}$ which also covers A in measure (denoted by the same). Then

$$|A| \leq \sum_j |A \cap \tilde{K}_j| \leq \mu |\cup_j \tilde{K}_j| \leq \mu |\cup_j (\tilde{K}_j \cup (\tilde{K}_j)^m)|.$$

Now we show that

$$|\cup_j (\tilde{K}_j \cup (\tilde{K}_j)^m)| \leq \frac{m+1}{m} |\cup_j (\tilde{K}_j)^m|. \tag{5.1.1}$$

Let E be the interior of $\cup_j (\tilde{K}_j)^m$ and consider the open bounded sets of the real line $E_x = \{t \in \mathbb{R} : (x,t) \in E\}$. Take the decomposition of E_x into a countable set of disjoint open intervals $E_x = \cup_j I_x^j$ and for every $I_x^j = (a, b)$ take $TI_x^j = (a - \frac{1}{m}(b-a), b)$. Define also $TE_x = \cup_j TI_x^j$ and $TE = \cup_x TE_x$, see

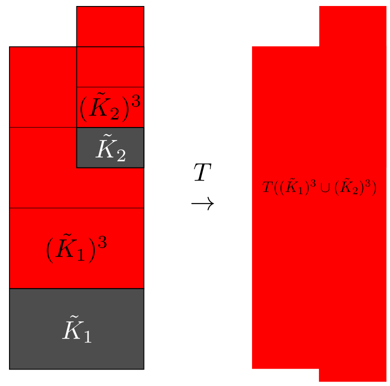


Figure 5.3: Transformation T.

figure 5.3. By Fubini,

$$\begin{aligned}
|TE| &= \int_{\mathbb{R}^n} |TE_x|, \\
&= \int_{\mathbb{R}^n} | \cup_j TI_x^j|, \\
&\leq \frac{m+1}{m} \int_{\mathbb{R}^n} \sum_j |I_x^j|, \\
&= \frac{m+1}{m} |E|, \\
&\leq \frac{m+1}{m} | \cup_j (\tilde{K}_j)^m|.
\end{aligned}$$

On the other hand, the interior of $\cup_j(\tilde{K}_j \cup (\tilde{K}_j)^m)$ is contained in TE, by the definition of T and the definition of the stacks. This concludes the proof. □

The following lemma uses the previous covering and shows some diminish of the measure of the level sets. However, it has the disadvantage that can not be iterated. The reason being that the larger level set is measured in $K_{1,1}$ meanwhile the smaller one is measured in $K_{1,d_m+1}(0, d_m)$. Still it will be an important step for the proof of Theorem 5.1.1.

Lemma 5.1.8. *Let $\sigma, m \geq 4, d_i, \mu_3, M_3$ as in Corollary 5.1.5 and $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \text{ in } C_{3^{\sigma(m-1)}(2\sqrt{n}),d_m+1}(0,d_m),$$

$$\inf_{\cup_{i=1}^m K_{3^i,3^{\sigma i}}(0,d_i)} u \leq 1.$$

Then, for any $i \in \mathbb{N}$ and for $\mu_m = \frac{m+1}{m}\mu_3$ and $M_m = M_3^m$, we have that,

$$\left|\left\{u \geq M_m^{i+1}\right\} \cap K_{1,1}\right| \leq \mu_m \left|\left\{u \geq M_m^{i}\right\} \cap K_{1,d_m+1}(0,d_m)\right|.$$

Proof. Let

$$A := \{u \geq M_m^{i+1}\} \cap K_{1,1}.$$

By Corollary 5.1.5 we know that,

$$|A| \leq |\{u \geq M_3\} \cap K_{1,1}| \leq \mu_3 |K_{1,1}|.$$

So we can apply Lemma 5.1.7 to A, with respect to the fraction μ_3, to find a covering $\{K_j\}$ that satisfies the conclusions of that Lemma. It suffices to show now that,

$$\bigcup_j (\tilde{K}_j)^m \subseteq \left\{u \geq M_m^i\right\} \cap K_{1,d_m}(0,d_m).$$

Assume by contradiction that,

$$\bigcup_j (\tilde{K}_j)^m \nsubseteq \left\{u \geq M_3^{mi}\right\} \cap K_{1,d_m}(0,d_m).$$

By the construction of $(\tilde{K}_j)^m$ we know that $\cup_j(\tilde{K}_j)^m \subseteq K_{1,d_m}(0,d_m)$. Given this, the previous hypothesis sais that there exists some dyadic box $K_j = K_{r,r^\sigma\tau}(x,t)$ such that:

1. $|A \cap K_j| > \mu_3 |K_j|$,

2. $\inf\limits_{(\tilde{K}_j)^m} u \leq M_m^i$.

The rescaling $(y,s) \mapsto (r^{-1}(y-x), r^{-\sigma}(s-t))$ sends K_j to $K_{1,\tau}$ and the stack $(\tilde{K}_j)^m$ to a subset of $\cup_{i=1}^m K_{3^i,3^{\sigma i}\tau}(0,d_i\tau)$. Then we apply Corollary 5.1.5 to the rescaled function,

$$\tilde{u}(y,s) = \frac{u(ry+x, r^\sigma s + t)}{M_m^i}.$$

The conclusion of Corollary 5.1.5 now contradicts that $|A \cap K_j| > \mu_3|K_j|$. $\square$

5.2 Proof of the Point Estimate Theorem 5.1.1

We have now all the tools to prove Theorem 5.1.1. The first step is to give a discrete version of it which will be proved by induction, for σ close to two.

Lemma 5.2.1 (Discrete Point Estimate)**.** *Let σ as in Lemma 5.1.2 and $u \geq 0$ satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \textit{ in } B_{2\sqrt{n}3^{\sigma(m-1)}} \times (0,d_m],$$
$$\inf_{K_{1,1}(0,17)} u \leq 1.$$

Then, for any $k \in \mathbb{N}$,

$$|\{u \geq M^k\} \cap K_{1,1}| \leq \mu^k |K_{1,1}|,$$

for $m \geq 4$, $\mu \in (0,1)$ and $M > 0$ universal to be fixed in the proof.

Proof. We start with $\mu = \frac{\mu_m+1}{2}$ and $M \geq M_m$ for m sufficiently large and prove the Lemma by induction. The case $k = 1$ holds by Corollary 5.1.5 with $\tau = 1$ if $d_m \geq 17$. Assume now by contradiction that there exists some k such that,

$$|\{u \geq M^{k+1}\} \cap K_{1,1}| > \mu^{k+1} \geq \mu |\{u \geq M^k\} \cap K_{1,1}|.$$

Let $\{K_j\}$ be the dyadic disjoint covering of $A = \{u > M^{k+1}\} \cap K_{1,1}$ with respect to the fraction μ_3. The proof of Lemma 5.1.8 tells us that,

$$\bigcup_j (\tilde{K}_j)^m \subseteq \left\{u \geq M_m^{k+1}\right\} \cap K_{1,d_m}(0, d_m).$$

Therefore,

$$\begin{aligned}
\mu^{k+1} &< |\{u \geq M_m^{k+1}\} \cap K_{1,1}|, \\
&\leq \mu_m(|\{u \geq M_m^k\} \cap K_{1,1}| + | \cup_j (\tilde{K}_j)^m \cap K_{1,d_m}(0, d_m)|), \\
&\leq \mu_m(\mu^k + | \cup_j (\tilde{K}_j)^m \cap K_{1,d_m}(0, d_m)|).
\end{aligned}$$

Then, we fix m sufficiently large such that $c := \frac{\mu - \mu_m}{\mu_m} = \frac{1-\mu_m}{2\mu_m} > 0$ and

$$c\mu^k \leq | \cup_j (\tilde{K}_j)^m \cap K_{1,d_m}(0, d_m)|.$$

It implies that the union of the stacks $\cup_j(\tilde{K}_j)^m$ has to go above time $c\mu^k$. So, there is a dyadic box $K_j = K_{r,r^\sigma\tau}(x,t)$, such that $r^\sigma\tau \geq c\mu^k/m$. It implies that there exists some N large enough such that $d_{kN}r^\sigma\tau \geq 17$. Now the idea is to apply Corollary 5.1.5 to get a contradiction.

We need to check that

$$K_{1,1}(0,17) \subseteq \bigcup_{i=1}^{Nk} K_{3^{\sigma i}r,3^{\sigma i}r^\sigma\tau}(x,t+d_i\tau) \tag{5.2.2}$$

On one hand we already know, by the assumption on N, that the union $\cup_{i=1}^{Nk} K_{3^{\sigma i}r,3^{\sigma i}r^\sigma\tau}(x,t+d_i r^\sigma\tau)$ is at least sufficiently tall to reach the time $t = 17$. On the other hand the following set remains inside $\cup_{i=1}^{Nk} K_{3^{\sigma i}r,3^{\sigma i}r^\sigma\tau}(x,t+d_i\tau)$,

$$\begin{aligned}&\{(y,s) \in \mathbb{R}^n \times (t, t+d_{NK}r^\sigma\tau] : (s-t) > 2^\sigma\tau|y-x|^\sigma\}\\&\supseteq K_{1,1}(0,17).\end{aligned}$$

See figure 5.4.

By applying a rescaled version of Corollary 5.1.5 we get $|\{u \geq M_3^{Nk}\} \cap K_j| \leq \mu_3|K_j|$ which contradicts $|\{u \geq M_m^{k+1}\} \cap K_j| > \mu_m|K_j|$ if $M \geq M_3^N$. $\square$

Proof of Theorem 5.1.1. By Corollary 4.2.3 we have already covered the cases where σ is away from two modulus some harmless rescaling, very similar to the one we are about to explain. Assume then that σ is sufficiently close to two such that we can apply the results of the previous sections.

Let $(x,t) \in C_{1,1}(0,1)$ such that $u(x,t) \leq 2\inf_{C_{1,1}(0,1)} u$. Given that the domain of Lemma 5.2.1 is universally fixed there exists a rescaling, not too

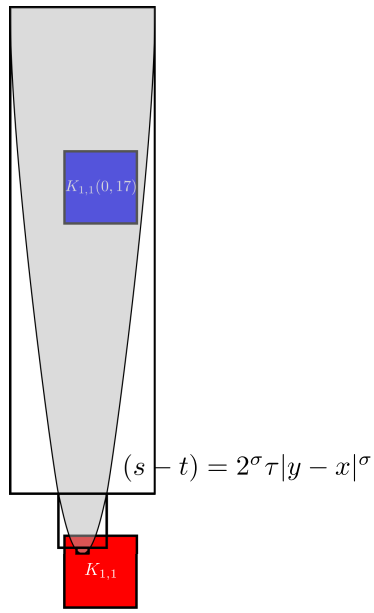


Figure 5.4: Covering $K_{1,1}(0,17)$ with a telescope.

large in magnitude, that send $(x,t) \mapsto (0,17)$ and $C_{1,1}$ to some set covering $K_{1,1}$. By applying Lemma 5.2.1 to,

$$\tilde{u}(y,s) := r^\sigma \frac{u(r(y-x), r^\sigma(s-t))}{\|f^+\|_\infty + \inf_{C_{1,1}(0,1)} u},$$

for some r not too small, we obtain,

$$\frac{|\{u > (\|f^+\|_\infty + \inf_{C_{1,1}(0,1)} u)M^k\} \cap C_{1,1}|}{|C_{1,1}|} \leq C\mu^k,$$

for some constant C uniform for σ close to two.

For any $\alpha > 0$ sufficiently large we consider the positive power k that makes

$$\left(\|f^+\|_\infty + \inf_{C_{1,1}(0,1)} u\right) M^k \leq \alpha < \left(\|f^+\|_\infty + \inf_{C_{1,1}(0,1)} u\right) M^{k+1}.$$

Then,

$$\mu^k = (M^k)^{(\ln M/\ln\mu)} \le \left(\frac{\alpha}{\|f^+\|_\infty + \inf_{C_{1,1}(0,1)} u}\right)^{(\ln M/\ln\mu)}$$

which gives us the desired estimate for $\varepsilon := -(\ln M/\ln\mu) > 0$. $\square$

5.3 Hölder Regularity

Finally, with the Point Estimate Theorem 5.1.1 at hand we can show a Hölder modulus of continuity of the solution as we did in Chapter 1 by using the Men Value formula.

Theorem 5.3.1 (Hölder regularity)**.** *Let $\sigma \in [1,2)$ $C_0 \ge 0$ and u satisfies,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \ge -C_0 \text{ in viscosity in } C_{1,1},$$

$$u_t - \mathcal{M}^+_{\mathcal{L}_0} u \ge C_0 \text{ in viscosity in } C_{1,1}.$$

Then, for every $(y,s),(x,t) \in C_{1/2,1/2}$,

$$\frac{|u(y,s)-u(x,t)|}{(|x-y|+|t-s|^{1/\sigma})^\alpha} \le C\left(\|u\|_{L^\infty(\bar{C}_{1,1})} + \|u\|_{L^\infty((-1,0]\mapsto L^1(\omega_\sigma))} + C_0\right),$$

for some universal $\alpha \in (0,1)$ and $C > 0$.

Corollary 5.3.2. *Let $\sigma \in [1,2)$, $f \in C(\Omega \times (a,b])$, I a uniformly elliptic operator and u satisfies,*

$$u_t - Iu = f \text{ in viscosity in } \Omega \times (a,b].$$

Then, for every $(y,s),(x,t) \in \Omega' \times (a',b] \subset\subset \Omega \times (a,b]$,

$$\frac{|u(y,s)-u(x,t)|}{(|x-y|+|t-s|^{1/\sigma})^\alpha} \le C\left(\|u\|_{L^\infty(\bar{\Omega}\times[a,b])} + \|u\|_{L^\infty((a,b]\mapsto L^1(\omega_\sigma))} + \|f + I0\|_{L^\infty(\Omega\times(a,b])}\right),$$

for some universal $\alpha \in (0,1)$ and $C > 0$ depending on the domains.

By scaling it suffices to show the following Diminish of Oscillation Lemma.

Lemma 5.3.3 (Diminish of Oscillation)**.** *Let u satisfies,*

$$\begin{aligned} u_t - \mathcal{M}^-_{\mathcal{L}_0} u &\geq -\eta && \text{in } C_{2,2},\\ u_t - \mathcal{M}^+_{\mathcal{L}_0} u &\leq \eta && \text{in } C_{2,2},\\ \|u\|_{L^\infty(C_{R,2})} &\leq 1/2, \\ \sup_{s\in(-2,0]} |u(y,s)| &\leq \left(\frac{|y|}{R}\right)^{1/2} && \text{in } \partial_p C_{R,2}, \end{aligned}$$

for some universal $\eta > 0$ sufficiently small and $R \geq 2$ sufficiently large. Then,

$$\operatorname*{osc}_{C_{1,1}} u \leq (1-\theta),$$

for some universal $\theta \in (0,1)$.

Remark 5.3.4. *In order to iterate this lemma to prove a Hölder modulus of continuity at the origin we need to see that the rescaling of the solution still satisfies the same hypothesis. This will impose further restrictions on how small $1/R$ and θ are. Assume without loss of generality that $u \geq -1/2+\theta$ in $C_{1,1}$ and consider then,*

$$\tilde{u}(y,s) := \frac{u(R^{-1}y, R^{-\sigma}t) - \theta/2}{1-\theta}.$$

The construction of $\tilde{u}$ is given so that $\|\tilde{u}\|_{L^\infty(C_{R,2})} \leq 1/2$. The equations get satisfied if at least $(1-\theta) \geq R^{-\sigma}$. Finally, the bound on the boundary data holds if $(1-2\theta) \geq R^{-1/2}$ which is the strongest restriction.

Proof. We consider two cases, either $|\{u \geq 0\} \cap C_{1,1}(0,-1)|$ is larger or smaller than $|C_{1,1}(0,-1)|/2$. Assume without loss of generality that the first case holds. Let,

$$v(y,s) = (u(y,s) + 1/2)^+.$$

$w := (u - v)$ is identically zero in $C_{3,2}$ and satisfies for $(x,t) \in C_{2,2}$,

$$\begin{aligned}(w_t - \mathcal{M}^- w)(x,t) &\geq - \inf_{K \in \mathcal{K}_0} \int_{B_3^c(-x)} u(y+x,t)K(y)dy,\\ &\geq -\frac{\Lambda(2-\sigma)}{R^{1/2}} \int_{B_{R-2}^c} \frac{|y|^{1/2}}{|y|^{n+\sigma}} dy.\end{aligned}$$

which can be made smaller than η for R sufficiently large, independently of $\sigma \in [1,2)$. Now by Theorem 3.2.3 we have that v satisfies the following in viscosity in $C_{2,2}$,

$$v_t - \mathcal{M}^-_{\mathcal{L}_0} v \geq (u_t - \mathcal{M}^-_{\mathcal{L}_0} u) - (w_t - \mathcal{M}^-_{\mathcal{L}_0} w) \geq -2\eta.$$

Applying Theorem 5.1.1 to the translated function $\tilde{v}(y,s) := v(y, s-1)$ we get that,

$$\frac{1}{2} \leq \frac{|\{\tilde{v} \geq 1/2\} \cap C_{1,1}|}{|C_{1,1}|} \leq C \left(\inf_{C_{1,1}(0,1)} \tilde{v} + \eta \right)^{\varepsilon}.$$

Finally we choose η sufficiently small to get $\inf_{C_{1,1}(0,1)} \tilde{v} \geq \theta := (2C)^{-1/\varepsilon}/2 > 0$. Going back to u, this implies that $u \geq -1/2 + \theta$ in $C_{1,1}$. $\square$

Proof of Theorem 5.3.1. It suffices to prove that the Hölder modulus of continuity holds at the origin and then complete the other cases by translating and scaling the following proof. Assume also, without loss of generality that for some $\tilde{\eta} \sim \eta$,

$$\|u\|_{L^\infty(\bar{C}_{1,1})} + \|u\|_{L^\infty((-1,0]\mapsto L^1(\omega_\sigma))} + \|f + I0\|_{L^\infty(C_{1,1})} \leq \tilde{\eta}.$$

Then, for the truncation $u_1 := \max(1/2, \min(-1/2, u))$ we know, as in the proof of Lemma 5.2.1, that u_1 satisfies in viscosity in $C_{1/2,1}$,

$$(u_1)_t - \mathcal{M}^-_{\mathcal{L}_0} u_1 \geq -C\tilde{\eta},$$
$$(u_1)_t - \mathcal{M}^+_{\mathcal{L}_0} u_1 \leq C\tilde{\eta}.$$

So this is how we fix $\tilde{\eta}$ such that $C\tilde{\eta} = \eta$.

Now we apply Lemma 5.2.1 to $u_2(y,s) := u_1(R^{-1}y, R^{-\sigma}s)$. This implies a diminish of oscillation of u_2 in $C_{1,1}$ which can then be iterated according to the Remark 5.3.4 to get a Hölder modulus of continuity at the origin for u_2 an finally u by pulling back the changes of variables. □

5.4 Applications: Estimates up to the boundary and Hölder regularity for the first derivatives

The barriers we gave on Chapter 2 combined with the previous interior regularity estimate also work to give us an estimate up to the boundary.

Theorem 5.4.1 (Regularity up to the boundary). *Let Ω be a domain with the exterior ball property and u satisfies*

$$u_t - \mathcal{M}^+_{\mathcal{L}_0} u \leq C_0 \quad \textit{in viscosity in } \Omega \times (a,b],$$
$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -C_0 \textit{ in viscosity in } \Omega \times (a,b].$$

Let ρ be a modulus of continuity such that

$$|u(x,t) - u(y,s)| \leq \rho(|x-y| \vee |t-s|)$$

for every $(x,t) \in (\partial\Omega \times (a,b]) \cup (\Omega \times \{a\})$ and $(y,s) \in \partial_p(\Omega \times (a,b])$. Then there is another modulus of continuity $\bar{\rho}$, depending on the domain and ρ, such that for every $(x,t) \in \bar{\Omega} \times [a,b]$ and $(y,s) \in \mathbb{R}^n \times [a,b]$.

$$|u(x,t) - u(y,s)| \leq \bar{\rho}(|x-y| \vee |t-s|).$$

Proof. This result goes back to the Corollaries 3.3.2 and 3.3.3 that we proved in order to get the existence of solutions taking the boundary and initial values in a continuous way. Combined with Theorem 5.3.1, they provide us with a modulus of continuity for $(x,t) \in \Omega \times (a,b]$ and $(y,s) \in \mathbb{R}^n \times [a,t]$. For $s \geq t$ then we only have to consider $y \in \Omega^c$, otherwise we just have to interchange the points. In this case let $x_0 \in \partial\Omega$ with $r = \operatorname{dist}(x,\partial\Omega) = |x - x_0|$. Then

$$\begin{aligned} |u(x,t) - u(y,s)| &\leq \bar{\rho}(r) + \rho(|x_0 - y| \vee (s-t)), \\ &\leq 2\bar{\rho}(|x-y| \vee (s-t)). \end{aligned}$$

This tell us how we need to modify the modulus of continuity to conclude the Theorem. □

For translation invariant operators it is know that, formally, the first derivatives also satisfy an equation in the same uniformly elliptic family. In principle, Theorem 5.3.1 should provides with an estimate for the first derivatives. However, the nonlocality has to be controlled by imposing a further restriction on the kernels which is what we do in the next definition.

Definition 5.4.1. *Let $\mathcal{L}_1 = \mathcal{L}_1(\rho, \sigma, \lambda, \Lambda, \beta) \subseteq \mathcal{L}_0(\rho, \sigma, \lambda, \Lambda, \beta)$ be the set of pairs $(K, b) \in \mathcal{L}_0$ such that,*

$$|DK(y)| \leq \frac{\Lambda}{|y|^{n+\sigma+1}}.$$

Theorem 5.4.2 (Hölder Regularity for the spatial gradient)**.** *Let I be translation invariant in space and uniformly elliptic with respect to $\mathcal{L}_1$, $f \in C^{0,1}_x C_t(\Omega \times (a, b])$ and u satisfies,*

$$u_t - Iu = f \text{ in viscosity in } \Omega \times (a, b].$$

Then $u \in C^{1,\alpha}_x(\Omega \times (a, b])$ and for every $(y, s), (x, t) \in \Omega' \times (a', b] \subset\subset \Omega \times (a, b]$,

$$\frac{|Du(x,t) - Du(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \leq C \left(\|u\|_{L^\infty(\Omega \times (a,b])} + \|u\chi_{\Omega^c}\|_{L^\infty((a,b] \mapsto L^1(\omega_\sigma))} \right.$$
$$\left. + \|f + I0\|_{C^{0,1}_x(\Omega \times (a,b])} \right),$$

for some universal $\alpha \in (0, 1)$ and $C > 0$ depending on the domains.

Proof. Assume without loss of generality that $\Omega \times (a, b] = C_{2,1}$, $\Omega' \times (a', b] = C_{1/2,1/2}$ and

$$\|u\|_{L^\infty(C_{2,1})} + \|u\chi_{B_2^c}\|_{L^\infty((-1,0] \mapsto L^1(\omega_\sigma))} + \|f + I0\|_{C^{0,1}_x(C_{2,1})} \leq 1.$$

As in the proof of Theorem 5.3.1, assume also that $u = 0$ in B_2^c, keeping the equation in $C_{1,1}$ but now with u globally bounded by a constant of order one.

Let $\bar{\alpha}$ the Hölder exponent obtained by Theorem 5.3.1 and assume that it is not the reciprocal of an integer by taking it smaller if necessary. Let $\delta = 1/(4\lfloor 1/\bar{\alpha} \rfloor)$. Fix a unit vector $e \in \mathbb{R}^n$, a number $h \in (0, \delta/8)$ and, for

$k = 1, 2, \ldots, \lfloor 1/\bar{\alpha} \rfloor$, let η^k be a smooth cut-off function supported in $B_{k+1/4} := B_{(3/4-(k+1/4))\delta}$ and equal to one in $B_{k+1/2} := B_{(3/4-(k+1/2))\delta}$. Define in this way the following incremental quotients,

$$
\begin{aligned}
w^{h,0} &:= u, \\
w^{h,k}(y,s) &:= \frac{u(y+he,s) - u(y,s)}{|h|^{\bar{\alpha}k}}, \\
w_1^{h,k}(y,s) &:= \frac{(\eta^k u)(y+he,s) - (\eta^k u)(y,s)}{|h|^{\bar{\alpha}k}}, \\
w_2^{h,k}(y,s) &:= \frac{((1-\eta^k)u)(y+he,s) - ((1-\eta^k)u)(y,s)}{|h|^{\bar{\alpha}k}}.
\end{aligned}
$$

We will show that, given $k \in \{0, 1, \ldots, \lfloor 1/\bar{\alpha} \rfloor - 1\}$ and $C_k := C_{3/4-k\delta, 3/4-k\delta}$ such that,

$$\|w^{h,k}\|_{L^\infty(C_k)} \leq A(k) \tag{5.4.3}$$

then in C_{k+1} a stronger estimate holds,

$$\frac{|w^{h,k}(x,t) - w^{h,k}(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \leq A(k+1) \tag{5.4.4}$$

where the constants $A(k)$ are independent of h. In order to do this we will analyze the equation for $w_1^{h,k}$.

$w_1^{h,k}$ is bounded by the hypothesis (5.4.3),

$$|w_1^{h,k}| \leq A(k) + \|\eta^k\|_{C^{0,\bar{\alpha}k}}.$$

By using that the equation is translation invariant we have that u and $u(\cdot + he, \cdot)$ satisfy the same equation in translated domains. By Theorem

3.2.3, $w^{h,k}$ satisfies the following inequalities in $C_{1,1}$ in the viscosity sense,

$$w_t^{h,k} - \mathcal{M}^-_{\mathcal{L}_1} w^{h,k} \geq -1,$$
$$w_t^{h,k} - \mathcal{M}^+_{\mathcal{L}_1} w^{h,k} \leq 1.$$

The function $w_1^{h,k}$ satisfies a similar equation as $w^{h,k}$ in $C_{k+3/4}$, the difference is on the right hand side introduced by the cutoff,

$$(w_1^{h,k})_t - \mathcal{M}^-_{\mathcal{L}_1} w_1^{h,k} \geq -1 + \mathcal{M}^-_{\mathcal{L}_1} w_2^{h,k},$$
$$(w_1^{h,k})_t - \mathcal{M}^+_{\mathcal{L}_1} w_1^{h,k} \leq 1 + \mathcal{M}^+_{\mathcal{L}_1} w_2^{h,k}.$$

For $x \in B_{k+3/4}$ the terms $|\mathcal{M}^\pm_{\mathcal{L}_1} w_2^h|$ are controlled by $\|u\|_\infty = 1$ by using the restriction on $\mathcal{L}_1$. Indeed, for $(K,b) \in \mathcal{L}_1$, $|y| \leq \delta/8$ we have that $w_2^{h,k}(x+y) = 0$ and $Dw_2^{h,k}(x) = 0$. Then, by the product rule (we omitted the variable t in the following estimate),

$$\begin{aligned}
|L_{K,b} w_2^{h,k}(x)| &= \left| \int w_2^{h,k}(x+y,t) K(y) dy \right|, \\
&= \left| \int_{B^c_{\delta/8}} \frac{(1-\eta^k)u(x+y+h) - (1-\eta^k)u(x+y)}{|h|^{\bar{\alpha}k}} K(y) dy \right|, \\
&= \left| \int_{B^c_{\delta/8}} (1-\eta^k)u(x+y)|h|^{1-\bar{\alpha}k} \frac{K(y) - K(y-h)}{|h|} dy \right|, \\
&\leq C.
\end{aligned}$$

We get then the equations for $w_1^{h,k}$ in $C_{k+3/4}$

$$(w_1^{h,k})_t - \mathcal{M}^-_{\mathcal{L}_1} w_1^{h,k} \geq -C,$$
$$(w_1^{h,k})_t - \mathcal{M}^+_{\mathcal{L}_1} w_1^{h,k} \leq C.$$

By applying Theorem 5.3.1 to $w_1^{h,k}$ rescaled from $C_{k+3/4}$ to C_{k+1} we conclude that for a constant $A(k+1)$ independent of h,

$$\frac{|w_1^{h,k}(x,t)-w_1^{h,k}(y,s)|}{(|x-y|+|t-s|^{1/\sigma})^\alpha} \leq A(k+1) \text{ for every } (x,t),(y,s) \in C_{k+1}.$$

This is equivalent to (5.4.4).

By Lemma 5.3 in [17] we get that (5.4.4) implies that $w^{h,k+1}$ is also bounded by a constant independent of h. Therefore we can apply this procedure up to obtaining that $u \in C_x^{0,1}(C_{3/4,3/4})$. Finally, by applying the previous argument one more time to the Lipschitz quotient we conclude the Theorem. □

For the time dependence we already know, from the counterexample in Chapter 1, that more than Lipschitz regularity may be false if the boundary data is not controlled in time, even for the fractional heat equation. Now he restriction has to be imposed on the boundary data and initial data.

Theorem 5.4.3 (Hölder regularity for the time derivative)**.** *Let I be translation invariant in time and uniformly elliptic, $f \in C_xC_t^{0,1}(\Omega\times(a,b])$ and u satisfies,*

$$\begin{aligned} u_t - Iu = f, &\text{ in viscosity in } \Omega\times(a,b],\\ u \in C_t^{0,1}&(\Omega^c\times[a,b]),\\ u(\cdot,a) \in C^{1,1}&(\Omega). \end{aligned}$$

Then for every $(x,t),(y,s) \in \Omega' \times (a',b] \subset\subset \Omega \times (a,b]$ we have

$$\frac{|u_t(x,t)-u_t(y,s)|}{(|x-y|+|t-s|^{1/\sigma})^\alpha} \leq C\left(\|u_t\|_{L^\infty(\Omega^c\times[a,b])} + \|\mathcal{M}^\pm_{\mathcal{L}_0} u(\cdot,a)\|_\infty \right. \\ \left. + \|f + I0\|_{C^{0,1}_t(\Omega\times(a,b])}\right)$$

for some universal $\alpha \in (0,1)$ and $C > 0$ depending on the domains.

Proof. Let

$$\varphi_\pm(y,s) := u(y,a) \pm M(s-a)$$
$$M := \|u_t\|_{L^\infty(\Omega^c\times[a,b])} + \|\mathcal{M}^\pm_{\mathcal{L}_0} u(\cdot,a)\|_\infty + \|f+I0\|_{C^{0,1}_t(\Omega\times(a,b])}.$$

The barriers are constructed so that,

$$\begin{aligned} (\varphi_+)_t - \mathcal{M}^+_{\mathcal{L}_0}\varphi_+ &\geq \|f+I0\|_{C^{0,1}_t(\Omega\times(a,b])} && \text{in viscosity in } \Omega\times(a,b], \\ (\varphi_-)_t - \mathcal{M}^-_{\mathcal{L}_0}\varphi_- &\leq -\|f+I0\|_{C^{0,1}_t(\Omega\times(a,b])} && \text{in viscosity in } \Omega\times(a,b], \\ \varphi_+ \geq u &\geq \varphi_- && \text{in } \partial_p(\Omega\times(a,b]). \end{aligned}$$

By the comparison principle $\varphi_+ \geq u \geq \varphi_-$ also in $\mathbb{R}^n \times [a,b]$. It implies that $|u(y,s)-u(y,a)| \leq M(s-a)$ which is the Lipschitz regularity of u at $t=a$.

To get Lipschitz regularity for $t > a$ we consider, for $(y,s) \in \mathbb{R}^n \times (a,b-h]$,

$$w(y,s) = u(y,s+h) - u(y,s).$$

From the previous considerations we know that $w(y,a) \leq Mh$. Also, since u is Lipschitz in time in $\partial_p(\Omega \times (a,b])$, we have the estimate $w \leq Mh$ in

$\partial_p(\Omega\times(a,b-h])$. By the translation invariance of I and the Lipschitz continuity of f in time, w satisfies,

$$\begin{aligned} w_t - \mathcal{M}^-_{\mathcal{L}_0} w &\geq -h\|f + I0\|_{C^{0,1}_t(\Omega\times(a,b])} \text{ in viscosity in } \Omega\times(a,b-h],\\ w_t - \mathcal{M}^+_{\mathcal{L}_0} w &\leq h\|f + I0\|_{C^{0,1}_t(\Omega\times(a,b])} \quad \text{in viscosity in } \Omega\times(a,b-h]. \end{aligned}$$

Once again we use the maximum principle to conclude that $Mh \geq w$ and obtain the Lipschitz continuity of u.

Consider now,

$$w(x,t) = \frac{u(x,t+h) - u(x,t)}{h}.$$

Thanks to the previous considerations we know that w is bounded in $\mathbb{R}^n \times (a,b-h]$, uniformly in h. Moreover, w satisfies

$$\begin{aligned} w_t - \mathcal{M}^-_{\mathcal{L}_0} w &\geq -\|f + I0\|_{C^{0,1}_t(\Omega\times(a,b])} \text{ in viscosity in } \Omega\times(a,b-h],\\ w_t - \mathcal{M}^+_{\mathcal{L}_0} w &\leq \|f + I0\|_{C^{0,1}_t(\Omega\times(a,b])} \quad \text{in viscosity in } \Omega\times(a,b-h]. \end{aligned}$$

We can use now Corollary 5.3.1 to conclude that $w \in C^\alpha(\Omega'\times(a',b])$ uniformly in h. Passing to the limit $h \to 0$ we conclude the Theorem. $\square$

Remark 5.4.4. *The proof in Theorem 5.4.3 works also in the case when the boundary data has a general modulus of continuity $\rho(\tau)$ in time. In the interior the solution will have a better modulus of continuity. It would be of the form $\tau^\alpha\rho(\tau)$ if $\limsup_{\tau\to0}\tau^{\alpha-1}\rho(\tau) > 0$ or it would imply a modulus of continuity for u_t of the form $\tau^{\alpha-1}\rho(\tau)$ if $\limsup_{\tau\to0}\tau^{\alpha-1}\rho(\tau) = 0$. See Lemma 5.6 in [17].*

5.5 Oscillation Lemma

The following is an improvement of the Corollary 3.2.2. For its statement consider,

$$\begin{aligned}
\varphi(y,s) = \varphi(|y|,s) &:= ((1+s)^{1/\sigma} - |y|)^{-(n+\sigma)},\\
P_r(x,t) &:= \{(y,s) \in \mathbb{R}^{n+1} : 0 \le t-s \le r^\sigma - |y-x|^\sigma\},\\
P_r &:= P_r(0,0).
\end{aligned}$$

Lemma 5.5.1 (Oscillation Lemma)**.** *Let u satisfies,*

$$\begin{aligned}
u_t - \mathcal{M}^+_{\mathcal{L}_0} u \le 1 \ \text{ in viscosity in } P_1,\\
\|u^+\|_{L^\infty((-1,0]\mapsto L^1(\omega_\sigma))} \le 1.
\end{aligned}$$

Then

$$u \le C\varphi \text{ in } P_1,$$

for some universal $C > 0$.

The applications in the rest of this chapter uses the following corollary.

Corollary 5.5.2. *Let u satisfies,*

$$u_t - Iu \le f, \text{ in viscosity in } \Omega \times (a,b].$$

Then for every $\Omega' \times (a',b] \subset\subset \Omega \times (a,b]$,

$$\sup_{\Omega' \times (a',b]} u \le C\left(\|u^+\|_{L^\infty((a,b]\mapsto L^1(\omega_\sigma))} + \|(f+I0)^+\|_\infty\right),$$

for some universal $C > 0$ depending on the domains.

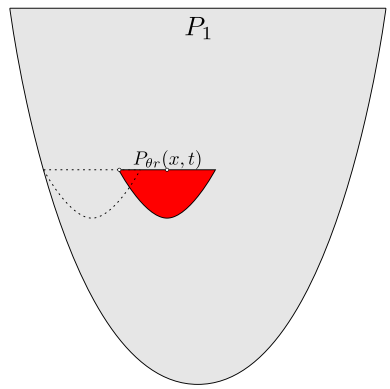


Figure 5.5: Closest point to to P_1^c in $P_{\theta r}(x,t)$.

Proof of Lemma 5.5.1. Let $M := \inf\{M' \in \mathbb{R}^+ : u \leq M'\varphi \text{ in } P_1\}$. It suffices to show that M is universally bounded.

Let $(x,t) \in P_1$ such that $u_0 := u(x,t) = M\varphi(x,t) = Md^{-(n+\sigma)}$ where $d := ((1+t)^{1/\sigma} - |x|) > 0$. Since $u^+ \in L^1(P_1)$, we have that, for any $\theta \in (0,1)$,

$$\left|\left\{u \geq \frac{u_0}{2}\right\} \cap P_{\theta d/8}(x,t)\right| \leq C\left|\frac{u_0}{2}\right|^{-1} \leq C\theta^{-(n+\sigma)}M^{-1}\left|P_{\theta d/8}(x,t)\right|.$$

As we want to get a contradiction if M is arbitrarily large, we will show that

$$\left|\left\{u < \frac{u_0}{2}\right\} \cap P_{\theta d/8}(x,t)\right| \leq \frac{1}{2}\left|P_{\theta d/8}\right|, \tag{5.5.5}$$

for some $\theta > 0$ to be fixed and M sufficiently large.

Let $r := d/2$. Over $P_{\theta r}(x,t)$, u is bounded from above by $M\varphi(|x|+\theta r, t) = u_0(1-\theta/2)^{-(n+\sigma)}$. Indeed, by the geometry of the paraboloids that we have constructed, it is not difficult to check that the closest point to P_1^c in $P_{\theta r}(x,t)$ gets realized at time t and at some point in $\partial B_{\theta r}(x)$. See figure 5.5.

Consider $w = (u_0(1-\theta/2)^{-(n+\sigma)} - u)^+$, which is equal to $(u_0(1-\theta/2)^{-(n+\sigma)} - u)$ in $P_{\theta r}(x,t)$. In the smaller domain $P_{\theta r/2}(x,t)$, w satisfies an equation with a right hand side that includes a contribution coming from the truncation,

$$w_t - \mathcal{M}^-_{\mathcal{L}_0} w \geq -1 - \mathcal{M}^+_{\mathcal{L}_0}(w+u) \text{ in viscosity in } P_{\theta r/2}(x,t)$$

Then we take a closer look at the second term for $(y,s) \in P_{\theta r/2}(x,t)$,

$$\begin{aligned}
\mathcal{M}^+_{\mathcal{L}_0}(w+u)(y,s) &\leq C \int_{B^c_{\theta r/2}} \frac{u^+(z+y,s)}{|z|^{n+\sigma}} dz, \\
&\leq C \left(\int_{\substack{(z+y,s)\in P_1 \\ z \in B^c_{\theta r/2}}} \frac{\delta_x(z-(\theta r/2)e_1)}{|z|^{n+\sigma}} dz \right. \\
&\left. + \int_{(z+y,s)\in P_1^c} \frac{u^+(z+y)}{|z|^{n+\sigma}} dz \right), \\
&\leq C(\theta r)^{-(n+\sigma)},
\end{aligned}$$

where δ_x is the delta distribution in space, such that we are using once again that $\|u^+\|_{L^1(P_1)} \leq 1$ an the worst case for the first integral is to concentrate the whole mass at some point in $\partial B^c_{\theta r/2}$. Therefore, for θ small, $(\theta r)^{-(n+\sigma)}$ becomes the leading term in the right hand side for the previous equation,

$$w_t - \mathcal{M}^-_{\mathcal{L}_0} w \geq -C(\theta r)^{-(n+\sigma)} \text{ in viscosity in } P_{\theta r/2}(x,t).$$

Now we apply the Point Estimate Theorem 5.1.1 to w with respect to the domains $P_{\theta r/2}(x,t)$ and $P_{\theta r/4}(x,t)$. To justify it you may use a standard covering argument. Then,

$$\begin{aligned}
\frac{\left|\{u < u_0/2\} \cap P_{\theta r/4}(x,t)\right|}{|P_{\theta r/4}(x,t)|} &= \frac{\left|\{w > u_0((1-\theta/2)^{-(n+\sigma)} - 1/2)\} \cap P_{\theta r/4}(x,t)\right|}{|P_{\theta r/4}(x,t)|}, \\
&\leq C(w(x,t) + (\theta r)^{-n})^{\varepsilon}(u_0((1-\theta/2)^{-(n+\sigma)} - 1/2))^{-\varepsilon}.
\end{aligned}$$

Now, by making $(1-\theta/2)^{-(n+\sigma)} \geq 3/4$ we get

$$\begin{aligned}\frac{\big|\{u < u_0/2\} \cap P_{\theta r/4}(x,t)\big|}{|P_{\theta r/4}(x,t)|} &\leq C(w(x,t) + (\theta r)^{-n})^{\varepsilon} u_0^{-\varepsilon},\\ &\leq C(((1-\theta/2)^{-(n+\sigma)} - 1) + M^{-1}\theta^{-n})^{\varepsilon}.\end{aligned}$$

Now we are almost done. We just have to fix θ even smaller such that $((1-\theta/2)^{-(n+\sigma)} - 1) \leq (1/2)(2C)^{-\varepsilon}$ which implies that for M sufficiently large the right hand side above becomes smaller than $1/2$. $\square$

This improvement of the maximum principle allows us to improve the previous Theorems 5.4.2 and 5.4.3. This idea was pointed out to us by Dennis Kriventsov.

Corollary 5.5.3. *Let I be translation invariant in space and uniformly elliptic with respect to $\mathcal{L}_1$, $f \in C^{0,1}_x C_t(C_{1,1})$ and u satisfies,*

$$u_t - Iu = f \text{ in viscosity in } C_{1,1}.$$

Then $u \in C^{1,\alpha}_x(C_{1/2,1/2})$ and for every $(y,s),(x,t) \in C_{1/2,1/2}$,

$$\frac{|Du(x,t) - Du(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^{\alpha}} \leq C\left(\|u\|_{C((-1,0]\mapsto L^1(\omega_\sigma))} + \|f + I0\|_{C^{0,1}_x(C_{2,1})}\right),$$

for some universal $\alpha \in (0,1)$ and $C > 0$.

For the time derivative we now only need to impose that,

$$\begin{aligned}\|u\|_{C^{0,1}((-1,0]\to L^1(\omega_\sigma))} &:= \|u\|_{L^\infty((-1,0]\to L^1(\omega_\sigma))} + [u]_{C^{0,1}((-1,0]\to L^1(\omega_\sigma))} < \infty,\\ [u]_{C^{0,1}((-1,0]\to L^1(\omega_\sigma))} &:= \sup_{\substack{[t-\tau,t]\subseteq(-1,0]\\ \tau\neq 0}} \frac{\|u(\cdot,t) - u(\cdot,t-\tau)\|_{L^1(\omega_\sigma)}}{\tau}.\end{aligned}$$

Corollary 5.5.4. *Let I be a translation invariant in time and uniformly elliptic, $f \in C_x C_t^{0,1}(\Omega \times (a,b])$ and u satisfies,*

$$u_t - Iu = f, \text{ in viscosity in } \Omega \times (a,b],$$

Then for every $(x,t),(y,s) \in \Omega' \times (a',b] \subset\subset \Omega \times (a,b]$ we have

$$\frac{|u_t(x,t) - u_t(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \leq C \left(\|u\|_{C^{0,1}((-1,0] \to L^1(\omega_\sigma))} + \|f + I0\|_{C_t^{0,1}(\Omega \times (a,b])} \right)$$

for some universal $\alpha \in (0,1)$ and $C > 0$ depending on the domains.

Chapter 6

Estimates by Approximation

The results of this section require us to go a little bit deeper in some functional analysis. First we extend the set of all non negative kernels $\mathbb{K}^+$ to the set of kernels, with no sign restriction, $\mathbb{K}$. $K \in \mathbb{K}$ means that $\delta\varphi(0,0;\cdot)K \in L^1(\mathbb{R}^n)$ for every test function φ at the origin. It implies, in particular, the integrability condition (2.1.2) that we saw at the beginning of Chapter 2 for K^+ and K^-.

Let $S \subseteq \mathbb{R}^{\mathbb{K}\times\mathbb{R}^n}$ be the set of all vectors $(l_{K,b})$ for which there exists a test function φ such that $L_{K,b}\varphi(0,0) = l_{K,b}$ for every $(K,b) \in \mathbb{K}\times\mathbb{R}^n$. This is analogous to the space of jets defined in the classical theory. To see how strong this restriction is, consider that $(l_{K,b}) \in S$ implies that $l_{2K,2b} = 2l_{K,b}$ for every coordinate $(K,b) \in \mathbb{K}\times\mathbb{R}^n$.

Given $\mathcal{L} \subseteq \mathbb{K}\times\mathbb{R}^n$, we may just define I as a function from $\Omega\times(a,b]\times(S\cap\mathbb{R}^{\mathcal{L}})$ to $\mathbb{R}$. The notion of (semi)continuity for I is defined in terms of the topology induced in S by the L^∞ norm of $\mathbb{R}^{\mathcal{L}}$. Uniform ellipticity, with respect to $\mathcal{L}\subseteq\mathcal{L}_0$, should be understood as before by the uniformly ellipticity identity (2.1.6) restricted to $S\cap\mathbb{R}^{\mathcal{L}}$.

Some results of this section can also be applied for operators I not necessarily elliptic. The notion of viscosity sub and super solution is the same as before, however, it might seem counter intuitive to say that $u_t - Iu \leq (\geq) f$ in viscosity whenever the function I is not monotone in $(l_{K,b})$ (degenerate ellipticity). The classical theory of viscosity solutions started in fact by studying Hamilton-Jacobi equations, which as we know are not elliptic. Lets us emphasize that this is still a meaningful definition, but we need to be careful to apply the results of the previous sections only when we can verify the required hypothesis.

One of the most important facts about S is that it is separable. To see why this is true notice that we just need to provide a countable set of test functions $\{\psi_i\}_{i\geq 1}$ such that for any arbitrary test function φ and any $\varepsilon > 0$ there exists some i such that,

$$\sup_{(K,b)\in\mathbb{K}\times\mathbb{R}^n} |L_{K,b}\varphi(0,0) - L_{K,b}\psi_i(0,0)| \leq \varepsilon.$$

This can be done by giving an enumeration of all test functions $\{\psi\} \subseteq C^\infty(\mathbb{R}^n \times \mathbb{R})$ with finitely many nonzero rational Fourier coefficients.

We define the norm for I and weak convergence by duality.

Definition 6.0.1 (Norm). *Given $I : \Omega \times (a,b] \times (S \cap \mathbb{R}^{\mathcal{L}}) \to \mathbb{R}$, let*

$$\|I\| = \|I\|_{\Omega\times(a,b],\mathcal{L}} := \sup\left\{\frac{|I(y,s,l_{K,b})|}{1+\|(l_{K,b})\|_\infty} : (y,s,(l_{K,b})) \in \Omega\times(a,b]\times(S\cap\mathbb{R}^{\mathcal{L}})\right\}.$$

This definition implies the standard Cauchy-Schwartz type of inequality,

$$|I\varphi(x,t)| \leq \|I\|(1 + |L_{K,b}\varphi(x,t)|).$$

The following is a useful Lemma at the time to estimate the norm of an operator.

Lemma 6.0.5. *Let,*

$$\mathcal{L} = \mathcal{L}(\sigma) := \left\{(K, b) \in \mathbb{K} \times \mathbb{R}^n : |K(y)| \leq |y|^{-(n+\sigma)},\ |b| \leq 1\right\}.$$

Given a test function φ, each one of the quantities

$$|D\varphi(0,0)|,\ \|\varphi(\cdot,0) - \varphi(0,0)\|_{L^1(\omega_\sigma)},\ \sup_{y \in B_1} \frac{|\delta\varphi(0,0;y)|}{|y|^2},$$

get bounded by some universal multiple of $\sup_{(K,b)\in\mathcal{L}} |L_{K,b}\varphi(0,0)|$. *Therefore,*

$$\begin{aligned} \|I\| \leq C \sup \Bigg\{ \frac{|I(x,t,L_{K,b}\varphi(x,t))|}{1+M} &: \varphi \text{ is a test function such that} \\ & |D\varphi(x,t)| \leq M, \\ & \sup_{y\in B_1} \frac{|\delta\varphi(x,t;y)|}{|y|^2} \leq M, \\ & \|\varphi(\cdot,t) - \varphi(x,t)\|_{L^1(\omega_\sigma)} \leq M \Bigg\}. \end{aligned}$$

Proof. The bound for $|D\varphi(0,0)| > 0$ follows by using $L_{K,b} = \frac{D\varphi(0,0)}{|D\varphi(0,0)|} \cdot D$. The bound for $\|\varphi(\cdot,0) - \varphi(0,0)\|_{L^1(\omega_\sigma)}$ follows by using L_K with $K := |y|^{-(n+\sigma)}\chi_{B_1^c}(y)$. The bound for $\sup_{y\in B_1} \frac{|\delta\varphi(x,t;y)|}{|y|^2}$ follows by using L_K with $K(y) := |y|^{-(n+\sigma)}\chi_{B_1}(y)$.

From the definition of S, we get that the supremum defining $\|I\|$ is taken with respect to $(x,t,l_{K,b}) = (x,t,L_{K,b}\varphi(x,t))$ among test functions φ and points $(x,t) \in \Omega \times (a,b]$. Given that M is the maximum of the three previously considered quantities, we still get that M is bounded by some universal multiple of $\sup_{(K,b)\in\mathcal{L}} |L_{K,b}\varphi(0,0)|$ and

$$\frac{|I(x,t,L_{K,b}\varphi(x,t))|}{1 + \sup_{(K,b)\in\mathcal{L}} |L_{K,b}\varphi(0,0)|} \leq C \frac{|I(x,t,L_{K,b}\varphi(x,t))|}{1+M}$$

which concludes the last part of the Lemma $\square$

Definition 6.0.2 (Weak convergence of operators)**.** *A sequence of continuous operators $\{I_i\}_{i\geq 1}$ converges weakly to an operator I in $\Omega \times (a, b]$ if for every test function $(\varphi, C_{r,\tau}(x, t))$, with $C_{r,\tau}(x, t) \subseteq \Omega \times (a, b]$, $I\varphi$ converges to $I\varphi$ locally uniformly in $C_{r,\tau}(x, t)$.*

From these definition we can easily see that convergence in norm implies weak convergence.

Weak convergence provide us with a stability theorem stronger than the one we already proved in Theorem 3.1.1. The proof goes along the same lines. Using the same notation as in such proof, the only modification that has to be done is to check that $\liminf_{i\to\infty} I_i\varphi_i(x_i, t_i) \geq I\varphi(x, t)$ which is provided by the weak convergence. In fact, the result would still hold for lower semicontinuous sequences of operators such that $\liminf_{i\to\infty} I_i\varphi(x_i, t_i) \geq I\varphi(x, t)$ for every $(x_i, t_i) \to (x, t^-)$.

Theorem 6.0.6 (Stability)**.** *Let $\{I_i\}_{i\geq 1}$ be a sequence of continuous operators, let $\{u_i\}_{i\geq 1}$ and $\{f_i\}_{i\geq 1}$ be sequences of functions such that:*

1. *$I_i \to I$ weakly in $\Omega \times (a, b]$,*

2. *$(u_i)_t - I_i u_i \geq f_i$ in the viscosity sense in $\Omega \times (a, b]$,*

3. *$u_i \to u$ in the Γ sense in $\Omega \times (a, b]$,*

4. *$\liminf_{i\to\infty} f_i(x_i, t_i) \geq f(x, t)$ for every $(x_i, t_i) \to (x, t^-)$ in $\Omega \times (a, b]$,*

Then $u_t - Iu \geq f$ in the viscosity sense in $\Omega \times (a,b]$.

The following property follows from the separability of S as in the proof of Banach-Alaoglu or Ascoli-Arzelà.

Property 6.0.1 (Compactness). *Every equibounded and equicontinuous set of functions from $\Omega \times (a,b] \times (S \cap \mathbb{R}^{\mathcal{L}})$ to $\mathbb{R}$ has a weak accumulation point with the same modulus of continuity.*

In particular, if $\{I_k\}_{k\geq 1}$ is a sequence of translation invariant, uniformly elliptic operators with respect to $\mathcal{L}$, such that $I_k 0 = 0$, then there exists a subsequence $\{I_{k_j}\}_{j\geq 1}$ that converges weakly to some operator I which is also uniformly elliptic with respect to $\mathcal{L}$.

6.1 Stability by Compactness

In the next result we obtain that a viscosity (sub and super) solution of operators $I_\pm$ sufficiently close in norm to a translation invariant, uniformly elliptic operator I remain close to the respective solution for I. Notice that no ellipticity requirement is necessary for the operators $I_\pm$.

Theorem 6.1.1 (Stability by compactness). *Let $\mathcal{L} \subseteq \mathcal{L}_0, \varepsilon > 0, M > 0, \alpha \in (0,\sigma)$ and ρ a modulus of continuity. There exists some $\eta > 0$ sufficiently small and $R > 0$ sufficiently large, depending on all the previous data, such that for any triplet of operators I, I_-, I_+ and any pair of functions u, v satisfying,*

1. *I is translation invariant, uniformly elliptic with respect to $\mathcal{L}$ and satis-*

fies $I0 = 0$.

2. $\|I_\pm - I\| \leq \eta$,

3. $u_t - Iu = 0$ *in viscosity in* $C_{1,1}$,

4. $v_t - I_- v \geq -\eta$ *and* $v_t - I_+ v \leq \eta$ *in viscosity in* $C_{1,1}$,

5. $u = v$ *in* $\partial_p C_{1,1}$,

6. $|u(y,s)| \leq M|y|^\alpha$ *for* $(y,s) \in B_1^c \times (-1,0]$

7. *For every* $(x,t),(y,s) \in ((\bar{B}_R \setminus B_1) \times [-1,0]) \cup (\bar{B}_R \times \{-1\})$.

$$|u(x,t) - u(y,s)| \leq \rho(|x-y| \vee |t-s|),$$

then,

$$|u - v| \leq \varepsilon \text{ in } C_{1,1}.$$

Proof. Assume by contradiction that the result does not hold for some data $\mathcal{L}, \varepsilon, M$ and ρ. Then there are a sequences R_k, η_k, I^k, I^k_-, I^k_+, u_k, v_k, such that $R_k \nearrow \infty$, $\eta_k \searrow 0$ and all the assumptions of the theorem are valid but $\sup_{C_{1,1}} |u_k - v_k| > \varepsilon$. We see now how to get a contradiction with uniqueness by using the Stability Theorem 3.1.1.

By the Boundary Regularity Theorem 5.4.1, we get that each u_k and v_k are equicontinuous in $C_{R_k,1}$. By Arzela-Ascoli we can extract a subsequence that converges locally uniformly to some functions u and v respectively. The bound given on the tails is important in order to say that, by dominated convergence, u_k and v_k also converge in $C((-1,0] \mapsto L^1(\omega_\sigma))$.

By the compactness property of the previous section we can also assume that I^k weakly converges to some operator I, translation invariant and uniformly elliptic with respect to $\mathcal{L}$. Because I^k_-, I^k_+ get closer and closer in norm to I^k we also get that they converge weakly to I.

Finally, by the Stability Theorem 3.1.1, u and v solve the same equation $w_t - Iw = 0$ with the same boundary data therefore, by the Maximum Principle Theorem 3.2.1, $u = v$ contradicting that $\sup_{C_{1,1}} |u_k - v_k| > \varepsilon$. $\square$

Remark 6.1.2. *Hypothesis 6 was required in order to have that the tails of u_k converge in $C((-1,0] \mapsto L^1(\omega_\sigma))$ which would not hold by assuming that $\|u\|_{C((-1,0] \mapsto L^1)} \leq M$. An even more general hypothesis can be worked out by using some fixed control as the one needed to apply the Kolmogorov compactness theorem.*

6.2 $C^{1,\alpha}$ Estimates by Approximation

As a consequence of the previous section we obtain the Cordes-Nirenberg type of estimate the we prove after the following Diminish of Oscillation Lemma. As we have already discussed in Chapter 1, the hypothesis have to strengthened, moreover in this case also the conclusion is strengthened in order to deal with the nonlocallity during the iteration.

Lemma 6.2.1 (Diminish of oscillation)**.** *Let I be a uniformly elliptic transla-*

tion invariant operator with respect to $\mathcal{L}_1$, such that the equation $v_t - Iv = 0$ has interior $C^{1,\alpha}$ estimates in space and time given that the boundary and initial data are regular enough (see Theorems 5.4.2 and Corollary 5.5.4). There exist some universal constants $\eta, \kappa, \theta > 0$ sufficiently small and $A, R > 1$ sufficiently large such that for any pair uniformly elliptic operators I_+ and I_- and u such that:

1. $\|I_\pm - I\| \leq \eta$,
2. $u_t - I_+u \leq \eta$ *and* $u_t - I_-u \geq -\eta$ *in the viscosity sense in* $C_{2R,2}$,
3. $\|u\|_{L^\infty(C_{1,1})} + \|u\|_{C^{0,1}([-1,0]\mapsto L^1(\omega_\sigma))} \leq 1$.
4. $|u(y,s)| \leq A|y|^{1+\alpha/2}$ *for* $(y,s) \in B_1^c \times (-1,0]$.

Then there exists some affine function $l(x) = a + b \cdot x$ with $|a|, |b| \leq A$, such that

$$\frac{|(u-l)(y,s)|}{\kappa(1-\theta)} \leq \frac{1 + \kappa^{-(1+\alpha/2)}|y|^{1+\alpha} + \kappa^{-\sigma}|s|}{3} \text{ for } (y,s) \in C_{1,1}.$$

In particular,

$$\sup_{C_{\kappa,\kappa^\sigma}} |u - l| \leq \kappa(1-\theta).$$

Remark 6.2.2. *We should ask ourselves how small should κ and θ be in order to be able to iterate the Lemma. We consider then $\tilde{u}$, given by*

$$\tilde{u}(y,s) := \frac{(u-l)(\kappa y, \kappa^\sigma s)}{\kappa(1-\theta)}.$$

The first three hypothesis hold if we assume $\kappa^{\sigma-1} \leq (1-\theta)$ and that the operators remain close at smaller scales, i.e.,

$$\|I - I_\pm\|_\sigma := \sup_{\kappa\in(0,1]} \left\|I^\kappa - I^\kappa_\pm\right\| \leq \eta,$$

where $I^\kappa_\pm$ and I^κ are rescalings of the previous operators,

$$\begin{aligned} I^\kappa(y, s, l_{K,b}) &:= I(\kappa y, \kappa^\sigma s, l_{K^\kappa,b^\kappa}),\\ K^\kappa(y) &:= \kappa^{n+\sigma} K(\kappa y),\\ b^\kappa &:= \kappa^{\sigma-1}\left(b + \int_{B_1\setminus B_\kappa} yK(y)dy\right), \end{aligned}$$

with similar definitions for $I^\kappa_\pm$.

The last hypothesis to control the tail have to be checked by splitting the domain in two regions. For $(y,s) \in (B_{\kappa^{-1}} \setminus B_1) \times (-1, 0]$ there is no additional restriction to ask. From the conclusion of the Lemma we get that,

$$|\tilde{u}(y, s)| \leq \frac{1 + \kappa^{\alpha/2}|y|^{1+\alpha} + |s|}{3} \leq \frac{2}{3}|y|^{1+\alpha/2} + \frac{\kappa^{\alpha/2}}{3}|y|^{1+\alpha} \leq A|y|^{1+\alpha/2}.$$

For $(y,s) \in (\mathbb{R}^n \setminus B_{\kappa^{-1}}) \times (-1, 0]$ we use the fact that the affine function is controlled in terms of A,

$$|\tilde{u}(y, s)| \leq A\left(\frac{\kappa^{\alpha/2}}{(1-\theta)}|y|^{1+\alpha/2} + \left(\frac{1}{\kappa(1-\theta)} + \frac{1}{(1-\theta)}|y|\right)\right).$$

The right hand side above is smaller than $A|y|^{1+\alpha/2}$ if $3\kappa^{\alpha/2} \leq (1-\theta)$.

Proof. We use Theorem 6.1.1 to approximate u by more regular solutions. The hypothesis of the Lemma verify also those of Theorem 6.1.1. In particular we have assumed that the equations get satisfied in $C_{2R,2}$ in order to have a universal Hölder modulus of continuity in $\bar{C}_{R,1}$. Given ε we can fix η and R such that there exists $h \in C^{1,\bar{\alpha}}(C_{1/2,1/2})$ such that,

1. $h_t - Ih = 0$ in viscosity in $C_{1,1}$,
2. $h = u$ in $\partial_p C_{1,1}$,
3. $a = h(0,0)$ and $b = D_x h(0,0)$ both bounded in norm by some universal A given by the interior estimates.
4. For $l(y) = a+b\cdot y$, $|(h-l)(y,s)| \leq A(|y|^{1+\bar{\alpha}}+|s|)$ for $(y,s) \in C_{1,1}$ also from the the interior estimates and the $C^{0,1}([-1,0] \mapsto L^1(\omega_\sigma))$ hypothesis on the boundary data.
5. $|h - u| \leq \varepsilon$ in $C_{1,1}$ from Theorem 6.1.1.

Notice that A, which is now fixed, is independent of ε.

By the triangular inequality $|(u-l)(y,s)| \leq \varepsilon + A(|y|^{1+\bar{\alpha}} + |s|)$ for $(y,s) \in C_{1,1}$. This already shows the diminish of oscillation required if we impose $3A \leq (1-\theta)\min(\kappa^{-\alpha/2}, \kappa^{-(\sigma-1)})$ and then fix $3\varepsilon = (1-\theta)\kappa$. □

Theorem 6.2.3 (Cordes-Nirenberg Estimate)**.** *For $\sigma > 1$, let I be a uniformly elliptic translation invariant operator with respect to $\mathcal{L}_1$, such that the equation $v_t - Iv = 0$ has interior $C^{1,\alpha}$ estimates in space and time given that the boundary and initial data are regular enough (see Theorems 5.4.2 and Corollary*

5.5.4). There exist some $\eta > 0$ sufficiently small such that for any pair of uniformly elliptic operators I_+, I_-, $f_+, -f_- \in USC(C_{1,1})$ and u such that:

1. $\|I - I_\pm\|_\sigma < \eta$,
2. $u_t - I_+ u \le f_+$ *and* $u_t - I_- u \ge -f_-$ *in the viscosity sense in* $C_{2R,2}$,
3. $u \in C^{0,1}([-1,0] \mapsto L^1(\omega_\sigma))$.

Then $u \in C^{1,\alpha}(C_{1/2,1/2})$ *and for every* $(x,t), (y,s) \in C_{1/2,1/2}$,

$$\frac{|Du(x,t) - Du(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \le C\left(\|u\|_{C^{0,1}([-1,0]\mapsto L^1(\omega_\sigma))} + \|f_+\|_\infty + \|f_-\|_\infty\right).$$

for some universal $\alpha \in (0,1)$ and $C > 0$ depending on the domains, uniform as $\sigma \to 2$ but degenerating as $\sigma \to 1$.

Proof. It suffices to show that Du has a Hölder modulus of continuity at the origin. Then it can be extended to $C_{1/2,1/2}$ by a standard covering argument.

A rescaling of u puts us into the hypothesis of the previous Lemma, in particular we are using the Corollary of the Oscillation Lemma 5.5.1 to bound u in $C_{1,1}$ by $\|u\|_{C^{0,1}([-1,0]\mapsto L^1(\omega_\sigma))}$. We choose κ, θ sufficiently small such that it can be iterated according to Remark 6.2.2. In this way we get a sequence of affine functions $l_k(y) = a_k + b_k \cdot y$ such that it verifies inductively, for $(y,s) \in C_{1,1}$ and $\delta := \kappa(1-\theta) < 1$,

$$\delta^{k+1} \ge u(\kappa^{k+1}y, \kappa^{\sigma(k+1)}s) - \left(\delta^k l_{k+1}(\kappa y) + \delta^{k-1} l_k(\kappa^2 y) + \ldots + l_1(\kappa^{k+1}y)\right).$$

u then approaches geometrically the following sequence of affine functions in the cylinders $C_{\kappa^k,\kappa^{\sigma k}}$,

$$L_{k+1}(y) := \delta^k l_{k+1}(\kappa^{-k}y) + \delta^{k-1}l_k(\kappa^{-(k-1)}y) + \ldots + l_1(y).$$

Given that $\delta < 1$ and $|a_k| \leq A$, the zero order term in L_k converges geometrically to some constant a. Given that $\delta/\kappa = (1-\theta) < 1$ and $|b_k| \leq A$, also the coefficient vector in front of y in L_k converges geometrically to some vector b. By the triangular inequality we see that u also approaches geometrically the affine function $L(x) = a + b \cdot x$ in the cylinders $C_{\kappa^k,\kappa^{\sigma k}}$. This proves the $C^{1,\alpha}$ modulus of continuity required at the origin. $\square$

As a Corollary we see that operators which are close to be a multiple of the fractional Laplacian inherit higher order regality estimates from the fractional heat equation.

Corollary 6.2.4. *For $\sigma > 1$, let I be translation invariant and uniformly elliptic with respect to $\mathcal{L}_1(\sigma,\rho,\Lambda-\eta,\Lambda)$, $f \in C^{0,1}(\Omega \times (a,b])$ and u satisfies,*

$$u_t - Iu = f, \text{ in viscosity in } \Omega \times (a,b],$$

Then, for η sufficiently small, we have that $Du, u_t \in C^{1,\alpha}_x C^\alpha_t(\Omega \times (a,b])$ and for every $(x,t),(y,s) \in \Omega' \times (a',b] \subset\subset \Omega \times (a,b]$ and $v = u_t$ or $v = Du$,

$$\frac{|Dv(x,t) - Dv(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \leq C\left(\|u\|_{C((-1,0]\to L^1(\omega_\sigma))} + \|f + I0\|_{C^{0,1}_x(\Omega\times(a,b])}\right)$$
$$\frac{|v_t(x,t) - v_t(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \leq C\left(\|u\|_{C^{0,1}((-1,0]\to L^1(\omega_\sigma))} + \|f + I0\|_{C^{0,1}_t(\Omega\times(a,b])}\right)$$

for some universal $\alpha \in (0,1)$ and $C > 0$ depending on the domains, uniform as $\sigma \to 2$ but degenerating as $\sigma \to 1$.

Proof. By Theorem 5.4.2 we know that $u \in C^{1,\alpha}_x(\Omega \times (a,b])$. For $i \in \{1,\dots,n\}$, let $u_i := Du \cdot e_i$ such that it satisfies,

$$
\begin{aligned}
(u_i)_t - \mathcal{M}^+_{\mathcal{L}_1} u_i &\leq \|f + I0\|_{C^{0,1}_x(\Omega\times(a,b])} \quad \text{in viscosity in } \Omega \times (a,b],\\
(u_i)_t - \mathcal{M}^-_{\mathcal{L}_1} u_i &\geq -\|f + I0\|_{C^{0,1}_x(\Omega\times(a,b])} \text{ in viscosity in } \Omega \times (a,b].
\end{aligned}
$$

By rescaling, we can assume that ρ is sufficiently large and β is sufficiently small. Let $K(y) := \Lambda(2-\sigma)|y|^{-(n+\sigma)}$, we will show that for η sufficiently small $\|L_K - \mathcal{M}^\pm_{\mathcal{L}_1}\|_\sigma$ is as small as needed in order to apply Theorem 6.2.3 and conclude the proof for the spatial gradient.

Given $\kappa \in (0,1]$, the rescalled operators become,

$$
\begin{aligned}
L^\kappa_K &= L_K,\\
(\mathcal{M}^\pm_{\mathcal{L}_1(\sigma,\rho,\Lambda-\eta,\Lambda,\beta)})^\kappa &= \mathcal{M}^\pm_{\mathcal{L}_1(\sigma,\kappa^{-1}\rho,\Lambda-\eta,\Lambda,\kappa^{\sigma-1}\beta)}.
\end{aligned}
$$

Then, for any test function φ,

$$
\begin{aligned}
0 &\leq (L_K - \mathcal{M}^+_{\mathcal{L}_1(\sigma,\kappa^{-1}\rho,\Lambda-\eta,\Lambda,\kappa^{\sigma-1}\beta)})\varphi(0,0),\\
&\leq (L_K - \mathcal{M}^-_{\mathcal{L}_1(\sigma,\kappa^{-1}\rho,\Lambda-\eta,\Lambda,\kappa^{\sigma-1}\beta)})\varphi(0,0),\\
&\leq (2-\sigma)\int \frac{(\eta\chi_{B_\rho}(y) + \Lambda\chi_{B^c_\rho}(y))\delta\varphi^+(0,0;y)}{|y|^{n+\sigma}}dy + \beta|D\varphi(0,0)|.
\end{aligned}
$$

Given that,

$$
\frac{|\delta\varphi(0,0;y)|}{|y|^2},\ \|\varphi(\cdot,0) - \varphi(0,0)\|_{L^1(\omega_\sigma)},\ |D\varphi(0,0)| \leq M
$$

then,

$$|(L_K - \mathcal{M}^{\pm}_{\mathcal{L}_1(\sigma,\kappa^{-1}\rho,\Lambda-\eta,\Lambda,\kappa^{\sigma-1}\beta)})\varphi(0,0)| \leq CM\left(\eta + \rho^{-\sigma} + \beta\right),$$

for some universal constant $C > 0$. This implies the claim for $\|L_K - \mathcal{M}^{\pm}_{\mathcal{L}_1}\|_\sigma$ by using the Lemma 6.0.5 and concludes the proof for the spatial derivatives of u.

The proof for u_t is similar to the previous one starting by using Theorem 5.4.3 instead. □

6.3 Regular Approximations

In this section we consider $\sigma > 1$ and I translation invariant and uniformly elliptic with respect to $\mathcal{L}_1$.

We will see that the Dirichlet problem being solved by u,

$$\begin{aligned} u_t - Iu &= f \text{ in viscosity in } C_{1,1}, \\ u &= g \text{ in } \partial_p C_{1,1}. \end{aligned} \tag{6.3.1}$$

can be approximated by a family of Dirichlet problems, parametrized by a small number $\varepsilon > 0$ and being solved by u_ε,

$$\begin{aligned} (u_\varepsilon)_t - I_\varepsilon u_\varepsilon &= f_\varepsilon \text{ in viscosity in } C_{1,1}, \\ u_\varepsilon &= g_\varepsilon \text{ in } \partial_p C_{1,1}, \end{aligned} \tag{6.3.2}$$

such that $I_\varepsilon \to I$ in norm and $f_\varepsilon, g_\varepsilon, u_e \to f, g, u$ respectively and locally uniformly. The most important fact about this approximation will be that we can construct the data $I_\varepsilon, f_\varepsilon$ and g_ε such that $Du_\varepsilon, (u_\varepsilon)_t \in C^{1,\alpha}_x C^\alpha_t(C_{1,1})$ for some $\alpha \in (0,1)$ depending on ε. This will allow us to recover in the next chapter estimates for viscosity solutions by assuming that u is sufficiently regular to begin with.

The general idea is to approximate I by I_ε, by modifying the domain of I, $\mathbb{R}^{\mathcal{L}_1}$ or more specifically $\mathcal{L}_1$. For each $(K,b) \in \mathcal{L}_1$ we will consider a new pair (K_ε, b) where K_ε replaces the kernel K by a multiple of the kernel for the fractional Laplacian in a small ball B_ε. We will see then that these new operators approach I as ε goes to zero and each one of them inherit the higher regularity estimates from the heat equation, although degenerating as $\varepsilon \to 0$.

Fix $\psi \in C^\infty(\mathbb{R}^+ \to [0,1])$ such that $\operatorname{supp}\psi = [0,3]$, $\psi = 1$ in $[0,1]$ and $|\psi'| \leq 1$. Let $\psi_\varepsilon(y) := \psi(\varepsilon^{-1}|y|)$ and for $(K,b) \in \mathcal{L}$ denote,

$$\begin{aligned}
K_\varepsilon(y) &:= K^0(y)\psi_\varepsilon(y) + K(y)(1-\psi_\varepsilon(y)),\\
K^0(y) &:= \frac{(2-\sigma)\Lambda}{|y|^{n+\sigma}},\\
\mathcal{L}_\varepsilon &:= \{(K_\varepsilon, b) \in \mathcal{L}_1 : (K,b) \in \mathcal{L}_1\}.
\end{aligned}$$

We define in this way $I_\varepsilon(l_{K,b}) := I(l_{K_\varepsilon,b})$ which is uniformly elliptic with respect to $\mathcal{L}_\varepsilon \subseteq \mathcal{L}_1$.

Lemma 6.3.1. *$\|I_\varepsilon - I\| \to 0$ as $\varepsilon \searrow 0$.*

Proof. By uniform ellipticity we get that for every $(l_{K,b}) \in (S \cap \mathbb{R}^{\mathcal{L}_1})$,

$$\begin{aligned} |(I - I_\varepsilon)(l_{K,b})| &\leq \sup_{(K,b)\in(S\cap\mathcal{L}_1)} |l_{K,b} - l_{K_\varepsilon,b}|, \\ &= \sup_{(K,b)\in(S\cap\mathcal{L}_1)} |l_K - l_{K_\varepsilon}|. \end{aligned}$$

Therefore we just have to estimate $|l_K - l_{K_\varepsilon}|$ uniformly in $(S \cap \mathcal{L}_1)$.

Let φ be a test function and $M > 0$ such that,

$$\sup_{y\in B_1} \frac{|\delta\varphi(0,0;y)|}{|y|^2} \leq M.$$

Then for every $(K,b) \in \mathcal{L}_1$,

$$\begin{aligned} |L_K\varphi(0,0) - L_{K_\varepsilon}\varphi(0,0)| &= \left| \int \delta\varphi(0,0;y) \left(K^0 - K\right)(y)\psi_\varepsilon(y)dy \right|, \\ &\leq CM\varepsilon^{2-\sigma}, \end{aligned}$$

for some constant C independent of $(K,b) \in \mathcal{L}_1$. Therefore we conclude, by using Lemma 6.0.5, that $\|I - I_\varepsilon\| \leq C\varepsilon^{2-\sigma}$. □

Let f_ε and g_ε be mollifications of f and g respectively. The Dirichlet problem (6.3.2) has a unique solution $u_\varepsilon \in C(\bar{C}_{1,1}) \cap C^{1,\alpha}(C_{1,1})$ by the Theorems 3.3.4, 5.4.1, 5.4.2 and 5.4.3. By Theorem 6.1.1, we further know that u_ε converges uniformly to the solution u of (6.3.1). Now we will show that, for fixed $\varepsilon \in (0,1)$, $v = (u_\varepsilon)_t$ or $v = (u_\varepsilon)_i$ is regular as in Corollary 6.2.4.

By Theorem 3.2.3, any choice of v satisfies,

$$\begin{aligned} v_t - \mathcal{M}^+_{\mathcal{L}_\varepsilon} v &\leq \|f + I0\|_{C^{0,1}(C_{1,1})} \quad \text{in viscosity in } C_{1,1}, \\ v_t - \mathcal{M}^-_{\mathcal{L}_\varepsilon} v &\geq -\|f + I0\|_{C^{0,1}(C_{1,1})} \text{ in viscosity in } C_{1,1}. \end{aligned}$$

For fixed $\varepsilon \in (0,1)$, let $(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa$ be the usual rescaling of $\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon}$,

$$
\begin{aligned}
(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa(l_{K,b}) &:= \pm \sup_{(K,b)\in\mathcal{L}_1} \left(\pm l_{K^\kappa_\varepsilon, b^\kappa}\right),\\
K^\kappa_\varepsilon(y) &:= \kappa^{n+\sigma} K_\varepsilon(\kappa y),\\
&= K^0(y)\psi_{\varepsilon\kappa^{-1}}(y) + K^\kappa(y)(1-\psi_{\varepsilon\kappa^{-1}}(y)),\\
K^\kappa(y) &:= \kappa^{n+\sigma} K(\kappa y),\\
b^\kappa &:= \kappa^{\sigma-1}\left(b + \int_{B_1\setminus B_\kappa} yK_\varepsilon(y)dy\right),\\
&= \kappa^{\sigma-1}\left(b + \int_{B_1\setminus B_\kappa} yK(y)(1-\psi_\varepsilon)(y)dy\right),\\
&= \kappa^{\sigma-1} b.
\end{aligned}
$$

Where the last equality holds if $\kappa \leq 3\varepsilon$. Then $v^\kappa(y,s) := \kappa^{-\sigma} v(\kappa y, \kappa^\sigma s)$ satisfies,

$$
\begin{aligned}
v^\kappa_t - (\mathcal{M}^{+}_{\mathcal{L}_\varepsilon})^\kappa v^\kappa &\leq \|f + I0\|_{C^{0,1}(C_{1,1})} \quad \text{in viscosity in } C_{\kappa^{-1},\kappa^{-\sigma}},\\
v^\kappa_t - (\mathcal{M}^{-}_{\mathcal{L}_\varepsilon})^\kappa v^\kappa &\geq -\|f + I0\|_{C^{0,1}(C_{1,1})} \text{ in viscosity in } C_{\kappa^{-1},\kappa^{-\sigma}}.
\end{aligned}
$$

In the limit, as $\kappa \searrow 0$, we should expect that $(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa$ converges to the coordinate function l_{K_0} which correspond to an operator which is just a multiple of the fractional Laplacian. This will provide us with higher regularity for u_ε by applying Theorem 6.2.3 to v^κ.

Lemma 6.3.2. *For fixed $\varepsilon \in (0,1)$, $\|(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa - l_{K^0}\| \to 0$ as $\kappa \searrow 0$. Therefore, $\|(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa - l_{K^0}\|_\sigma$ can be made arbitrarily small for κ sufficiently small.*

Proof. By the definitions of $(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa$ and l_{K^0} we get that for every $(l_{K,b}) \in$

$(S \cap \mathbb{R}^{\mathcal{L}})$,

$$|((\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa - l_{K^0})(l_{K,b})| \leq \sup_{(K,b)\in(S\cap\mathcal{L}_1)} |l_{K^\kappa_\varepsilon,b^\kappa} - l_{K^0}|.$$

Therefore we just have to estimate $|l_{K^\kappa_\varepsilon,b^\kappa} - l_{K^0}|$ uniformly in $(S \cap \mathcal{L}_1)$.

Let φ be a test function and $M > 0$ such that,

$$\max\left(|D\varphi(0,0)|, \|\varphi(\cdot,t) - \varphi(0,0)\|_{L^1(\omega_\sigma)}\right) \leq M.$$

Then, for $k \leq \varepsilon$,

$$\begin{aligned} |L_{K^\kappa_\varepsilon,b^\kappa}\varphi(0,0) - L_{K^0}\varphi(0,0)| &\leq \left|\int \delta\varphi(0,0;y)\left(K_0 + K^\kappa\right)(y)(1-\psi_{\varepsilon\kappa^{-1}})(y)dy\right| \\ &\quad + \beta M\kappa^{\sigma-1}, \\ &\leq CM(\varepsilon^{-\sigma}\kappa^\sigma + \kappa^{\sigma-1}), \end{aligned}$$

for some constant C independent of $(K,b) \in \mathcal{L}_1$. Therefore we conclude, by using Lemma 6.0.5, that $\|(\mathcal{M}^{\pm}_{\mathcal{L}_\varepsilon})^\kappa - l_{K^0}\| \leq C(\varepsilon^{-\sigma}\kappa^\sigma + \kappa^{\sigma-1})$. □

Summarizing, the consequence of this Lemma for v^κ is that $v^\kappa \in C^{1,\alpha}_x C^\alpha_t(C_{1,1})$. This concludes the main goal of this section.

Chapter 7

Concave equations

We have already seen that solutions of translation invariant equations with $\sigma = 1$ are as classical as they can be, recall the Theorems 5.4.2 and 5.4.3. In this last chapter we show that solutions of translation invariant, concave or convex equations with zero right hand side satisfy that all the linear operators applied to them are bounded. Which means they are a bit less than classical. In order to recover an Evans-Krylov type of Theorem as in [13] we would need to exploit an interplay between the Point Estimate Theorem 5.1.1 and the Oscillation Lemma 5.5.1. However the Oscillation Lemma we proved does not seem strong enough at this moment in order to be able to do that.

Definition 7.0.1. *Let $\mathcal{L}_2 = \mathcal{L}_2(\rho, \sigma, \lambda, \Lambda, \beta) \subseteq \mathcal{L}_1(\rho, \sigma, \lambda, \Lambda, \beta)$ be the set of pairs $(K, b) \in \mathcal{L}_1$ such that,*

$$|D^2 K(y)| \leq \frac{\Lambda}{|y|^{n+\sigma+2}}.$$

For this part we fix $\sigma > 1$, $\mathcal{L} \subseteq \mathcal{L}_2$, and u such that,

$$u_t - \mathcal{M}^-_{\mathcal{L}} u = 0 \text{ in viscosity in } C_{7,5},$$
$$\|u\|_{C^{0,1}((-5,0] \mapsto L^1(\omega_\sigma))} \leq 1.$$

By Section 6.3 we can assume that u is a classical solution with smooth boundary and initial data. Otherwise, we approximate u by a sequence of classical solutions with smooth boundary and initial data and recover the estimates of this chapter in the limit. The only thing we need to be careful about is that those a priori estimates are independent of (fractional) derivatives of u which are not accessible by the viscosity solutions.

The following general properties follow from the definitions of $\mathcal{M}^{\pm}_{\mathcal{L}}$. Some of them have already been used in previous discussions and we recall them as they will be extensively used. The notion of convolution we use is,

$$v * w(x) := \int w(x+y)v(y)dy.$$

Property 7.0.1. *Let $\alpha \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $\eta \in L^1(\mathbb{R}^n)$. Then the following holds for any regular function v,*

1. ***Homogeneity:*** $(\alpha v)_t - \mathcal{M}^-_{\mathcal{L}}(\alpha v) = \alpha\left(v_t - \mathcal{M}^-_{\mathcal{L}} v\right)$,
2. ***Translation invariance:*** $(b \cdot Dv)_t - \mathcal{M}^+_{\mathcal{L}}(b \cdot Dv) \leq b \cdot D\left(v_t - \mathcal{M}^-_{\mathcal{L}} v\right)$,
3. ***Concavity:*** $(\eta * v)_t - \mathcal{M}^-_{\mathcal{L}}(\eta * v) \leq \eta * \left(v_t - \mathcal{M}^-_{\mathcal{L}} v\right)$.

Corollary 7.0.3. *For $(K, b) \in \mathbb{K}^+ \times \mathbb{R}^n$ and $\varphi \in C^\infty_0(B_2 \mapsto [0,1])$ it holds that,*

$$(L_{K,b}u)_t - \mathcal{M}^+_{\mathcal{L}}(L_{K,b}u) \leq ([(1-\varphi)K] * u)_t - \mathcal{M}^-_{\mathcal{L}}([(1-\varphi)K] * u) \text{ in } C_{5,5}.$$

In particular, if $\operatorname{supp} K \subseteq B_1$ and $\varphi = 1$ in B_1, then,

$$(L_{K,b}u)_t - \mathcal{M}^+_{\mathcal{L}}(L_{K,b}u) \leq 0 \text{ in } C_{5,5}.$$

Proof. Let, for $\varepsilon \in (0,1)$, $K_\varepsilon := \chi_{B_\varepsilon^c} K$. We decompose the operator $L_{K_\varepsilon,b}$ as a sum of a local and a nonlocal operator, the nonlocal being the one appearing on the right hand side of the conclusion of the Lemma,

$$\begin{aligned} L_{K_\varepsilon,b} &= L + NL, \\ &:= (L_{K,b} - K(1-\varphi)*) + K(1-\varphi)*, \\ &= \left(\varphi K_\varepsilon * - \|K_\varepsilon\|_1 - \left(\int_{B_1} y K_\varepsilon(y)dy - b\right) \cdot D\right) + K(1-\varphi) * . \end{aligned}$$

Then,

$$\begin{aligned} (L_{K_\varepsilon,b}u)_t - \mathcal{M}^+_{\mathcal{L}}(L_{K_\varepsilon,b}u) &\leq \left((Lu)_t - \mathcal{M}^+_{\mathcal{L}}(Lu)\right) + \left((NLu)_t - \mathcal{M}^-_{\mathcal{L}}(NLu)\right), \\ &\leq L\left(u_t - \mathcal{M}^-_{\mathcal{L}} u\right) + \left((NLu)_t - \mathcal{M}^-_{\mathcal{L}}(NLu)\right). \end{aligned}$$

In $C_{5,5}$ the first term is zero as the local operator L does not take into account the values of $\left(u_t - \mathcal{M}^+_{\mathcal{L}} u\right)$ outside of B_7. The result now follows in the limit as $\varepsilon \searrow 0$ by the Stability Theorem 6.0.6. □

Property 7.0.2 (Integration by parts)**.** *Let $(K,b) \in \mathbb{K}^+ \times \mathbb{R}^n$ and $(\bar{K}(y), \bar{b}) := (K(-y), -b)$. Then the following holds for any pair of regular functions v and w such that at least one of them go to zero at infinity,*

$$\int v L_{K,b} w = \int w L_{\bar{K},\bar{b}} v.$$

In particular,

$$L_{K,b}(v * w) = v * (L_{K,b} w) = (L_{\bar{K},\bar{b}} v) * w.$$

Corollary 7.0.4. *For $(K,b) \in \mathcal{L}$ it holds that,*

$$(L_{K,b}u)_t - \mathcal{M}^+_{\mathcal{L}}(L_{K,b}u) \leq C \text{ in } C_{5,5},$$

for some universal constant $C > 0$.

Proof. Corollary 7.0.3 tells us that it suffices to estimate $([(1-\varphi)K] * u)_t - \mathcal{M}^-_{\mathcal{L}}([(1-\varphi)K] * u)$ in $C_{5,5}$,

$$\begin{aligned}
([(1-\varphi)K] * u)_t &= [(1-\varphi)K] * u_t,\\
&\leq C\|u\|_{C^{0,1}((-5,0]\mapsto L^1(\omega_\sigma))},\\
&= C,\\
\mathcal{M}^-_{\mathcal{L}}([(1-\varphi)K] * u) &= \inf_{(K',b')\in\mathcal{L}_2} L_{K',b'}([(1-\varphi)K] * u),\\
&= \inf_{(K',b')\in\mathcal{L}_2} (L_{\bar{K}',\bar{b}'}[(1-\varphi)K] * u),\\
&\geq -C.
\end{aligned}$$

In the last inequality we have used that $|DK(y)| \leq \Lambda'|y|^{-(n+\sigma+1)}$, $|D^2K(y)| \leq \Lambda'|y|^{-(n+\sigma+2)}$ and $\|u\|_{L^\infty((-5,0]\mapsto L^1(\omega_\sigma))} \leq 1$. □

From now on we fix, for $r_1 > r_2 > 0$, $\psi_{r_1,r_2} \in C^\infty_0(B_{r_1} \to [0,1])$ such that $\psi_{r_1,r_2} = 1$ in B_{r_2}.

Corollary 7.0.5. *Let $5 \geq r_1 > r_2 > 0$, $(K,b) \in \mathbb{K}^+ \times \mathbb{R}^n$ such that either $(K,b) \in \mathcal{L}$ or the following three hypothesis hold:*

1. $|b| \leq \beta'$,

2. $\operatorname{supp} K \subseteq B_1$,

3. $K(y) \leq (2-\sigma)\Lambda'|y|^{-(n+\sigma)}$.

Then,

$$(\psi_{r_1,r_2} L_{K,b} u)_t - \mathcal{M}^+_{\mathcal{L}}(\psi_{r_1,r_2} L_{K,b} u) \leq C \text{ in } C_{r_2,5},$$

for some universal constant $C > 0$ depending also on r_1, r_2, β' and Λ'

Proof. We use either Corollary 7.0.3 or 7.0.4 to get that $\psi_{r_1,r_2} L_{K,b} u$ satisfies in $C_{r_2,5}$,

$$\begin{aligned}(\psi_{r_1,r_2} L_{K,b} u)_t - \mathcal{M}^+_{\mathcal{L}}(\psi_{r_1,r_2} L_{K,b} u) &\leq C + \sup_{(K',b')\in\mathcal{L}_2} L_{K',b'}((1-\psi_{r_1,r_2})L_{K,b}u),\\ &= C + \sup_{(K',b')\in\mathcal{L}_2} K' * ((1-\psi_{r_1,r_2})L_{K,b}u)\end{aligned}$$

Now we take a closer look at $[K' * ((1-\psi_{r_1,r_2})L_{K,b}u)](x,t)$ for $(x,t) \in C_{r_2,5}$,

$$\begin{aligned}[K' * ((1-\psi_{r_1,r_2})L_{K,b}u)](x,t) &= [(K'(1-\psi_{r_1,r_2}(x+\cdot))) * L_{K,b}u](x,t),\\ &= [L_{\bar{K},\bar{b}}(K'(1-\psi_{r_1,r_2}(x+\cdot))) * u](x,t),\\ &\leq C.\end{aligned}$$

In the last inequality we have used that $|DK'(y)| \leq \Lambda|y|^{-(n+\sigma+1)}$, $|D^2K'(y)| \leq \Lambda|y|^{-(n+\sigma+2)}$ and $\|u\|_{C^{0,1}((-5,0]\mapsto L^1(\omega_\sigma))} \leq 1$ in order to obtain a bound independent of $(K',b') \in \mathcal{L}_2$. $\square$

7.1 Estimate for $\Delta^{\sigma/2}u$

Lemma 7.1.1. *Let $(K,b) \in \mathbb{K}^+$ such that:*

1. $|b| \le \beta$,
2. $K(y) \le (2-\sigma)\Lambda|y|^{-(n+\sigma)}$.

Then

$$|L_K u| \le C \text{ in } C_{1,1},$$

for some universal constant C.

Proof. We do it in several steps. Here is a summary of the strategy:

1. For $(K,b) \in \mathcal{L}$, we bound $L_{K,b}u$ from below by using the equation for u and the control we have for u_t inside the domain.
2. For $(K,b) \in \mathcal{L}$, we integrate by parts to have a control over $\|L_{K,b}u\|_{L^1(\omega_\sigma))}$ and then apply the Corollary of the Oscillation Lemma 5.5.1 to bound $L_{K,b}u$ from above.
3. For general (K,b), we use a tool coming from Fourier techniques to have a control over $\|L_{K,b}u\|_{L^1(\omega_\sigma))}$ and then apply the Corollary of the Oscillation Lemma 5.5.1 to bound $L_{K,b}u$ from above.
4. For general (K,b), we apply the previous step to $(K'',b'') = \Lambda(K',b') - \lambda(K,b)$ with $(K',b') \in \mathcal{L}$ to bound $L_{K,b}u$ from below.

Step 1: $(K,b) \in \mathcal{L}$, then $L_{K,b}u \geq -C$ in $C_{7,4}$.

It follows from the equation for u and Corollary 5.5.4,

$$L_{K,b}u \geq \mathcal{M}^-_{\mathcal{L}}u = u_t \geq -C\|u\|_{C^{0,1}((-5,0]\mapsto L^1(\omega_\sigma))}.$$

Step 2: $(K,b) \in \mathcal{L}$, then $L_{K,b}u \leq C$ in $C_{3,3}$.

We will use the Corollary of the Oscillation Lemma 5.5.1 applied to the truncation $\psi_{5,4}L_{K,b}u$. By Corollary 7.0.5, $\psi_{5,4}L_{K,b}u$ satisfies,

$$(\psi_{5,4}L_{K,b}u)_t - \mathcal{M}^+_{\mathcal{L}}(\psi_{5,4}L_{K,b}u) \leq C \text{ in } C_{4,4}.$$

We estimate now $\|\psi_{5,4}L_{K,b}u\|_{L^\infty((-4,0]\mapsto L^1(\omega_\sigma))}$. As $\psi_{5,4}L_{K,b}u$ is bounded from below and equal to zero outside B_5 it follows by integrating by parts,

$$\begin{aligned}\|\psi_{5,4}L_{K,b}u\|_{L^\infty((-4,0]\mapsto L^1(\omega_\sigma))} &\leq C\left(1 + \sup_{t\in(-4,0]}\left|\int \psi_{5,4}L_{K,b}u\right|\right),\\ &\leq C\left(1 + \|u\|_{L^\infty(C_{5,4})}\int_{B_1}|L_{\bar{K},\bar{b}}\psi_{5,4}|\right),\\ &\leq C.\end{aligned}$$

By the Corollary of the Oscillation Lemma 5.5.1, we can say now that $\psi_{5,4}L_{K,b}u$ gets universally bounded from above in $C_{3,3}$ where it coincides with $L_{K,b}u$.

Step 3: $(K,b) \in \mathbb{K}^+ \times \mathbb{R}^n$ such that

1. $|b| \leq \beta'$,
2. $K(y) \leq (2-\sigma)\Lambda'|y|^{-(n+\sigma)}$.

Then $L_{K,b}u \le C$ in $C_{1,1}$.

Once again, we will apply the Corollary of the Oscillation Lemma 5.5.1 to the truncation $\psi_{3,2}L_{K_l,b}u$, where $K_l = K\chi_{B_1^c}$. By Corollary 7.0.5, $\psi_{3,2}L_{K_l,b}u$ satisfies,

$$(\psi_{3,2}L_{K_l,b}u)_t - \mathcal{M}^-_{\mathcal{L}}(\psi_{3,2}L_{K_l,b}u) \le C \text{ in } C_{2,2}.$$

We estimate now $\|\psi_{3,2}L_{K_l,b}u\|_{L^\infty((-2,0]\mapsto L^1(\omega_\sigma))}$ by using an analogous result to the Theorem 4.3 in [13] for translation invariant linear equations. As in [13] it can be proved by Fourier techniques,

$$\begin{aligned}\|\psi_{3,2}L_{K_l,b}u\|_{L^\infty((-2,0]\mapsto L^1(\omega_\sigma))} &\le C\|L_{K_l,b}u\|_{L^\infty((-2,0]\mapsto L^2(B_2))},\\ &\le C\|L_{K',b'}u\|_{L^\infty((-3,0]\mapsto L^2(B_3))}.\end{aligned}$$

Where (K',b') is an arbitrary pair in $\mathcal{L}$. By the Corollary of the Oscillation Lemma 5.5.1, $\psi_{3,2}L_{K_l,b}u$ gets bounded from above in $C_{1,1}$ where it coincides with $L_{K_l,b}u$. Completing the kernel only adds the tail of u, which is controlled,

$$\begin{aligned}L_K u &= L_{K_l}u + L_{K\chi_{B_1^c}}u,\\ &\le C(1 + \|u\|_{L^\infty((-1,0]\mapsto L^1(\omega_\sigma))}).\end{aligned}$$

Step 4: $(K,b) \in \mathbb{K}^+$ such that:

1. $|b| \le \beta$,

2. $K(y) \le (2-\sigma)\Lambda|y|^{-(n+\sigma)}$.

Then $L_{K,b}u \geq -C$ in $C_{1,1}$.

Consider $(K', b') \in \mathcal{L}$ and $(K'', b'') := \Lambda(K', b') - \lambda(K, b)$ such that:

1. $|b''| \leq (\Lambda + \lambda)\beta$,

2. $K''(y) \leq (2 - \sigma)\Lambda^2 |y|^{-(n+\sigma)}$

From the second step, it suffices to show that $L_{K'',b''}u \geq -C$ in $C_{1,1}$. This follows from the third step. □

Corollary 7.1.2. *There is a universal constant $C > 0$ such that,*

$$(2 - \sigma) \int \frac{|\delta u(x, t; y)|}{|y|^{n+\sigma}} dy \leq C \text{ in } C_{1,1}.$$

In particular, by Morrey estimates, we have that $u \in C_x^{1,1-\alpha}(C_{1,1})$ for every $\alpha \in [1, \sigma)$, see [36].

Proof. Using $K(y) := \Lambda(2 - \sigma)|y|^{-(n+\sigma)}$ in the previous Lemma we get,

$$(2 - \sigma) \int \frac{\delta u(x, t; y)}{|y|^{n+\sigma}} dy \geq -C \text{ in } C_{1,1}.$$

Fixing $(x, t) \in C_{1,1}$ and using $K(y) := \Lambda(2-\sigma) \operatorname{sign}(\delta u(x, t; y))|y|^{-(n+\sigma)}$ in the previous Lemma we get,

$$(2 - \sigma) \int \frac{\delta^+ u(x, t; y)}{|y|^{n+\sigma}} dy \leq C.$$

Adding them up we conclude the Corollary. □

Part II

Free Boundaries on two Dimensional Cones

Chapter 8

Introduction

In this part we study a free boundary on a singular manifolds. The by now classical problem involves studying minimizers of the functional

$$J(u,\Omega) = \int_{\Omega} |Du|^2 + \chi_{\{u>0\}} \tag{8.0.1}$$

with a predetermined non negative boundary data. This situation appears in cavitational problems, flame propagation, optimal insulation among other models referenced for instance in the book [9]. The Euler-Lagrange equation gives the following over determined problem for u,

$$\Delta u = 0 \text{ in } \{u > 0\} \cap \Omega,$$
$$|Du^+| = 1 \text{ in } \partial\{u > 0\} \cap \Omega.$$

The regularity of u and its free boundary $\partial\{u > 0\}$ was obtained by L. Caffarelli and H. Alt in [1]. More complicated situations appear in the two phase problem where E also penalizes the set where u is negative, in which case the boundary data can have arbitrary sign and regularity estimates become more delicate, see [14–16]. Arbitrary metrics with some regularity condition are considered by the series of papers by Sandro Salsa and Fausto Ferrari [19, 23–25]. See also [20] for an alternative and elegant approach.

In this work we look at two phase problems with degenerate metrics. In terms of existence, the minimization problem can be solved in the functional space H^1 over manifolds with minimal assumptions of smoothness, for instance with corners. Our first attempt is to study the simplest case we could imagine, a **two dimensional cone generated by a smooth simple closed curve** γ **on the unit sphere**. The main question of interest is to study the interaction of the free boundary with the vertex.

The free boundary in the one phase problem given in (8.0.1) behaves similarly to minimal surfaces. For instance, it is well known that there are no nontrivial area minimizing cones in dimensions $n \leq 7$ while the Simons cone in dimension $n = 8$ is area minimizing. Similarly, there are no minimizing cone solutions to (8.0.1) in dimensions $n = 2, 3$ (see [18]) while in dimension $n = 7$ a minimizing cone does exist (see [21]) which is analogous to the Simons cone. In this work we provide another connection between minimal surfaces and the free boundary arising from (8.0.1). For distance minimizing geodesics on two dimensional cones (generated by a smooth simply connected curve γ on the sphere) the following proposition is well-known

Proposition 8.0.3. *If $l = length(\gamma) < 2\pi$, no distance minimizing geodesics pass through the vertex. If $l = length(\gamma) \geq 2\pi$, then there are distance minimizing geodesics that pass through the vertex.*

The proof when $l < 2\pi$ can be found in Section 4-7 in the book [22].

In this work we prove the analogous result of Proposition 8.0.3 for minimizers of (8.0.1) on a cone.

Theorem 8.0.4. *Let u be a minimizer of* (8.0.1). *If $l < 2\pi$, then the vertex $0 \notin \partial\{u = 0\}$. If $l \geq 2\pi$ the free boundary can pass through the vertex.*

It is worth noting that Theorem 8.0.4 also bears resemblance to the result obtained by H. Shahgholian in [34] where the free boundary in the obstacle problem can enter into the corner of a fixed boundary if and only if the aperture of the corner is greater than or equal to π. Many of the techniques and methods developed in studying the classical problem (8.0.1) aided in the study of the obstacle problem. The results and techniques of this work may aid in the future study of obstacle problems over rough obstacles which has applications in mathematical finance [32].

8.1 Outline

In Chapter 9 we discuss existence, regularity and stability of the minimizers. The proofs of many of these statements are simple adaptations from the arguments found in the classical literature and are left for an appendix chapter at the end. Chapter 10 is our main contribution. There we prove that, in the case $l < 2\pi$, the free boundary of our minimizers always avoid the vertex. Our approach consists in reducing the problem to find a better competitor against 1-homogeneous minimizers. Finally in Chapter 11 we discuss the situation when $l \geq 2\pi$. We provide some examples where the vertex belongs to the free boundary and for even larger values of l we also show that more than one positive phase can meet at the vertex.

Chapter 9

Preliminaries

We fix, without loss of generality, our two dimensional cone $\mathcal{C} \subseteq \mathbb{R}^3$ to have its vertex at the origin. Such a cone $\mathcal{C}$ embedded in $\mathbb{R}^3$ is a ruled surface that inherits a flat metric. By this we mean that for every open set $U \subseteq \mathcal{C} \setminus \{0\}$ there is always a local isometry that maps it to an open set of $\mathbb{R}^2 \setminus \{0\}$ with the flat metric. This follows from a parametrization of $\mathcal{C}$ given by polar coordinates, since $\mathcal{C} \cap B_1$ is just a one dimensional smooth simple closed curve that can be parametrized by arc length. In order to also have an injective isometry we can lift the previous map to the universal covering of $\mathbb{R}^2 \setminus \{0\}$ which we denote by $\mathcal{R}^2$. In polar coordinates $\mathcal{R}^2$ gets parametrized by a radius and an angle $(r, \theta) \in \mathbb{R}^+ \times \mathbb{R}$.

Let l be the length of the trace of the given cone in the unit sphere. From now on we just say that $\mathcal{C}$ has length l. This length gives us a canonical representation of $\mathcal{C} \setminus \{0\}$ as $\mathcal{R}^2/\{\theta \in l\mathbb{Z}\}$ with the flat metric in $\mathcal{R}^2$. We denote by ϕ_l the (isometry) quotient map going from $\mathcal{R}^2$ to $\mathcal{R}^2/\{\theta \in l\mathbb{Z}\}$.

A way to visualize what we have described so far is by cutting the cone by one of its rays starting at the origin and laying the surface flat, keeping in mind the identification at the boundary. In the case that $l < 2\pi$ it looks like

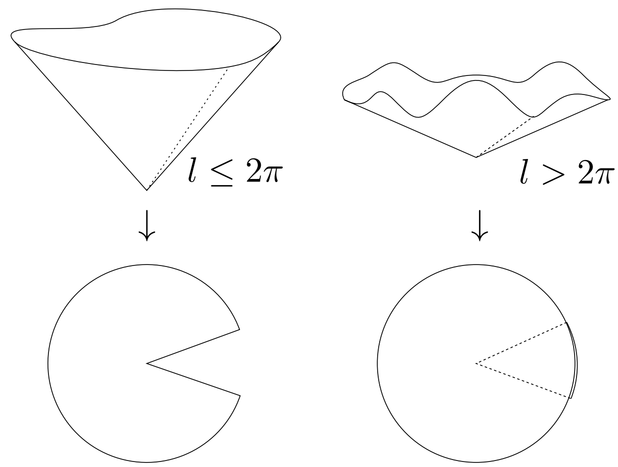


Figure 9.1: Cutting two cones and laying them flat

$\mathbb{R}^2$ minus a cone and in the case that $l > 2\pi$ we will have some overlap. See Figure 9.1.

Notice that all we have said so far also holds for any two dimensional cone embedded in $\mathbb{R}^n$ with $n \geq 2$. After looking at the universal covering of such cone minus its vertex the domain gets fixed to a quotient of the form $\mathcal{R}^2/\{\theta \in l\mathbb{Z}\}$.

9.1 Harmonic functions on a Cone

In this section we study some basic properties of harmonic functions on $\mathcal{C}$. Given the previous discussion, we have that for $f : \Omega \subseteq \mathcal{C} \to \mathbb{R}$ we can define any differential operator acting on f in the distributional sense. A test function $\varphi(x)$ in this case is a smooth function in $\mathcal{C} \setminus \{0\}$ with all its derivatives uniformly bounded in x and such that $\varphi(x)$ is continuous at the vertex. Notice that, because there is not a tangent plane at the vertex, we can not make sense of the gradient of a function at the vertex. However we can always ask if the function has a modulus of continuity even at the vertex.

The following proposition gives some equivalent definitions for subharmonic functions. We omit the proof.

Proposition 9.1.1 (Subharmonic functions)**.** *For a function $h : \Omega \subseteq \mathcal{C} \to \mathbb{R}$ the following are equivalent and in such cases we say that h is subharmonic:*

1. *For every $K \subseteq \Omega$ compact, $h \in H^1(K)$ and it minimizes the Dirichlet energy $\int_K |Dh|^2$ over all the functions less or equal than h in K and with the same boundary data as h in ∂K. Here we denoted by Df the tangential gradient and the integral is taken with respect to the area form in $\mathcal{C}$.*

2. *$h \in L^1(\Omega)$ and it has the mean value property for subharmonic functions in Ω.*

3. *Seeing as a function $h : \Omega' \subseteq \mathcal{R}^2/\{\theta \in l\mathbb{Z}\} \to \mathbb{R}$ (where $\Omega \setminus \{0\}$ gets mapped to Ω' by the isometry) $\Delta h \geq 0$ in Ω' in the sense of distributions and has the mean value property for subharmonic functions at the origin if $0 \in \Omega$.*

The definition of superharmonic functions is analogous to the previous one by changing the corresponding inequalities. A harmonic function is one which is both sub and super harmonic simultaneously. This definition in particular allows one to perform integration by parts and recover Green's formula even in the case where the vertex belongs to the domain of integration.

The following proposition follows from the Fourier series representation and gives us already some intuition about how differently harmonic functions behave according to the length of the cone. We also omit its proof.

Proposition 9.1.2. *Let $\mathcal{C}$ be a cone with length l. Any harmonic function h on $\mathcal{C}$ may be written as*

$$h(r,\theta) = \sum_{k=0}^{\infty} r^{2\pi k/l} \left(a_k \cos \frac{2\pi k}{l}\theta \; + \; b_k \sin \frac{2\pi k}{l}\theta \right). \tag{9.1.1}$$

In particular, $h \in C^{2\pi/l}$.

9.2 Minimization problem

Here we give the explicit minimization problem we want to study and show existence in H^1.

Given a domain $\Omega \subseteq \mathcal{C}$ and $\lambda_+ \neq \lambda_-$ non negative numbers, let $J : H^1(\Omega) \to \mathbb{R}$ given by,

$$J(u) = J(u, \Omega, \lambda_+, \lambda_-) = \int_\Omega |Du|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}}$$

The same proof given to show existence of minimizers of J with a given boundary data also applies to our case. Here is the proposition and its proof can be adapted from the one in [2].

Proposition 9.2.1. *Given $g \in H^1(\Omega)$ such that $J(g) < \infty$ there exists a minimizer $u \in H^1(\Omega)$ of J such that $u - g \in H^1_0(\Omega)$.*

Given u a minimizer of J, over the domain $\mathcal{C}_1 \sim B_1 \cap (\mathcal{R}^2/\{\theta \in l\mathbb{Z}\})$, the pull back $\tilde{f} = f \circ \phi_l^{-1}$ is also a minimizer of J over any compact set $\tilde{K} = \phi_l^{-1}(K)$. Most of the observations that can be made about the minimization problem posed in a domain in $\mathbb{R}^2$ can also be made about domains of the cone. Next we recall some of them.

First of all, since the functional J is not convex, minimizers are not necessarily unique.

There is the possibility, when the boundary data is large enough, that minimizers stay positive in the whole domain and therefore the Euler Lagrange equations say that the solution has to be harmonic. Notice that in such case the regularity of the solution degenerates as l becomes larger (see Proposition 9.1.2). More interesting cases arise when there is a phase transition. This occurs, for example, if the boundary data changes sign or if it is sufficiently small.

The Euler Lagrange equation associated with the minimization problem looks exactly the same at every point of the domain which is different from the vertex. We have to introduce some notation before giving the set of equations. Let $u^{\pm}$ the positive and negative parts of $u = u^{+} - u^{-}$ and $\Omega^{\pm} = \Omega \cap \{u^{\pm} > 0\}$.

Then

$$
\begin{aligned}
\Delta u = 0 \qquad & \text{in } \{u \neq 0\} \cap (\Omega \setminus \{0\}), \\
|Du^+| = \lambda_+ \qquad & \text{in } (\partial\Omega^+ \setminus \partial\Omega^-) \cap (\Omega \setminus \{0\}), \\
|Du^-| = \lambda_- \qquad & \text{in } (\partial\Omega^- \setminus \partial\Omega^+) \cap (\Omega \setminus \{0\}), \\
|Du^+|^2 - |Du^-|^2 = \lambda_+^2 - \lambda_-^2 \ & \text{in } (\partial\Omega^+ \cap \partial\Omega^-) \cap (\Omega \setminus \{0\}).
\end{aligned}
$$

What happens at the origin is actually the main concern of this work. Something that we can say is that if $u(0) \neq 0$ then u is also harmonic at the origin. The next interesting case is when $0 \in (\partial\Omega^+ \cup \partial\Omega^-)$.

9.3 Further properties of minimizers

In this section we comment on some of the fundamental properties of minimizers of J. Their proof are simple adaptations of the classical proofs given in [2, 40] and we leave them for the appendix of this part. Specifically we will discuss:

1. Initial regularity. For any cone we show that minimizers are at least Hölder continuos depending on l and the H^1 norm of the minimizer.

2. Stability of minimizers by uniform convergence.

3. Optimal regularity when $l \leq 2\pi$.

4. Compactness and 1-homogeneity of sequences of blow-ups when $l \leq 2\pi$.

9.3.0.1 Initial regularity

Initially we can use the results from [2] to say that the minimizer u is $C^{0,1}$ in every compact $\tilde{K}$ of the form $\tilde{K} = \phi_l^{-1}(K)$ and therefore also locally in $\Omega \setminus \{0\}$. In particular, the Lipschitz estimates in [2] are scale invariant and therefore in our situation it gives us a Lipschitz estimate that degenerates towards the vertex.

Proposition 9.3.1. *Given u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ then for $r \in (0, 1/2)$,*

$$\|Du\|_{L^\infty(\mathcal{C}_{1/2} \setminus \mathcal{C}_r)} \leq C r^{-1},$$

for some universal $C > 0$.

The next step is to check that u also remains continuous up to the vertex. In this sense we can show the following Theorem.

Theorem 9.3.2. *Let u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ then for any $\alpha \in (0, \min(1, 2\pi/l))$ we have that $u \in C^{\alpha/4}(\mathcal{C}_{1/10})$ with,*

$$|u(x) - u(y)| \leq C|x - y|^{\alpha/4} \text{ for every } x, y \in \mathcal{C}_{1/10},$$

and some universal $C > 0$.

Corollary 9.3.3. *Let u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$, then $u^\pm$ are continuous subharmonic functions satisfying $\Delta u^\pm = 0$ in $\mathcal{C}_1^\pm$.*

9.3.0.2 Stability

In the previous part we saw that for a minimizer u, the positive and negative parts $u^{\pm}$ are automatically subharmonic continuous functions and we even have a modulus of continuity for them. The stability of minimizers by uniform convergence depends on uniform equicontinuity and non degeneracy estimates. This allow us to say that if two minimizers are uniformly close then their zero sets are also close in the Hausdorff metric.

Theorem 9.3.4 (Stability)**.** *Let $\{u_k\}$ be a sequence of minimizers of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with λ_+ and λ_- different from zero converging to a function u in $\mathcal{C}_1$ with respect to the H^1 norm. Then:*

1. *$\{u_k\}$ also converges uniformly to u in $\mathcal{C}_{1/2}$,*

2. *Each one of the sets $\{u_k > 0\} \cap \mathcal{C}_{1/2}$ and $\{u_k < 0\} \cap \mathcal{C}_{1/2}$ converge to the respective set $\{u > 0\} \cap \mathcal{C}_{1/2}$, $\{u < 0\} \cap \mathcal{C}_{1/2}$ with respect to the Hausdorff distance,*

3. *u is also a minimizer of J.*

9.3.0.3 Optimal regularity

The optimal regularity expected for this problem can not be better than Lipschitz as in the classical case. On the other hand harmonic functions defined over cones with length $l > 2\pi$ may not be Lipschitz. Here we focus mainly on the case when $l \leq 2\pi$ in order to obtain the optimal regularity for the minimizers of J.

Theorem 9.3.5 (Optimal regularity when $l \leq 2\pi$)**.** *Let $l \leq 2\pi$, u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ and $\{u = 0\} \cap \mathcal{C}_{1/2} \neq \emptyset$; then for every $x_0 \in \mathcal{C}^+_{1/4}$,*

$$|Du(x_0)| \leq C,$$
$$|u(x_0)| \leq C \operatorname{dist}(x_0, \partial\mathcal{C}^+_1 \cap \mathcal{C}_{1/2}).$$

In the case $l > 2\pi$ we can still can ask ourselves if the minimizer u remains Lipschitz up to the vertex if $u(0) = 0$. This is the case for instance of problems with one phase. This follows from the observation that away from the origin the problem inherits the regularity from the classical case, therefore the gradient along the free boundary is constant independently of how close we get to the origin. In the case of having two phases there might be still some balance between the positive and negative phase that allows the gradient to grow to infinity as we approach the vertex. However we suspect that when $\lambda^+ \neq \lambda^-$ this is not the case.

9.3.0.4 Blows-up

As a consequence of the stability and the optimal regularity we obtain that a sequence of Lipschitz dilations of a given minimizer of J and centered at the origin, have an accumulation point which is also a minimizer J over any compact set of the cone. Moreover, by proving a monotonicity formula as in [40] we obtain that such an accumulation point is a 1-homogeneous function.

Corollary 9.3.6 (Blow-up limits)**.** *Let $l \leq 2\pi$ and u be a minimizer of $J =$*

$J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $u(0) = 0$ and $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$. For any sequence of blow-up $u_k = r_k^{-1} u(r_k \cdot)$ with $r_k \to 0$ we have that there exist an accumulation point $u \in C^{0,1}_{loc}(\mathcal{C})$ such that:

1. *u is also a minimizer of $J(K, \lambda_+, \lambda_-)$ for any compact set $K \subseteq \mathcal{C}$,*

2. *u is a 1-homogeneous function in $\mathcal{R}^2 / \{\vec{\theta} \in l\mathbb{Z}\}$.*

Chapter 10

The vertex and the free boundary: Case $l < 2\pi$

In this chapter we show that if $\mathcal{C}$ is a cone with length $l < 2\pi$, then $0 \notin (\partial\Omega^+ \cup \partial\Omega^-)$ for any minimizer u. The idea is to reduce the problem to 1-homogeneous minimizers by using Corollary 9.3.6.

Theorem 10.0.7. *Let $l < 2\pi$ and u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$. Then $0 \notin (\partial\Omega^+ \cup \partial\Omega^-)$.*

We split the proof into several Lemmas.

Lemma 10.0.8. *Let $l < 2\pi$ and u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$. Then $0 \notin (\partial\Omega^+ \cap \partial\Omega^-)$.*

Proof. Suppose by way of contradiction that $0 \in (\partial\Omega^+ \cap \partial\Omega^-)$. By Corollary 9.3.6 we have that there exists a limiting blow up u_0 which is homogeneous of order one. But homogeneous harmonic functions of order one are linear and then $\mathcal{H}^1(\{u_0 > 0\} \cap \partial B_1) = \mathcal{H}^1(\{u_0 < 0\} \cap \partial B_1) = \pi$. This is a contradiction with $l < 2\pi$. $\square$

Remark 10.0.9. *Lemma 10.0.8 coupled with the compactness and stability results from Section 2 works to show that there exists some $\varepsilon > 0$ such that the* *same result holds for $l < \pi - \varepsilon$. We won't discuss this proof here as this result is contained in the following Lemmas.*

The previous Lemma reduces the problem to study only cases with just one phase. From now on we will assume without lost of generality that $\lambda_+ = 1, \lambda_- = 0$ and the minimizers are non negative. Also, from the previous blow-up argument applied now to solutions with just one phase we can reduce the problem to showing that the function $v = x_2^+$ is not a minimizer of $J(K)$ for any compact set $K \subseteq \mathcal{C}$.

When we talk about the function x_2^+ defined in $\mathcal{C}$ we mean the following: Because $l < 2\pi$ there is an isometry

$$\phi : \mathcal{C} \setminus \{0\} \to \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : -\cot(l/2)|x_1| < x_2\}.$$

It is in this coordinate system that we define the function x_2^+. In the next section we will use that for $u : K \subset\subset \mathcal{C} \to \mathbb{R}$, the functional $J(u, K)$ can also be computed from $\tilde{u} = u \circ \phi^{-1}$ and $\tilde{K} = \phi^{-1}(K)$ in the following way,

$$J(u, K) = \tilde{J}(\tilde{u}, \tilde{K}) = \int_{\tilde{K}} |D\tilde{u}|^2 + |\{\tilde{u} > 0\} \cap \tilde{K}|.$$

Notice that a competitor v_0 for v in K such that $\{v_0 > 0\} \cap K \subseteq \{v > 0\} \cap K$ gives that $\tilde{J}(\tilde{v}_0, \tilde{K}) > \tilde{J}(v, \tilde{K})$ because v is the unique minimizer of $\tilde{J}(\tilde{K})$ with its boundary data. Therefore, if we want to find competitor with smaller values of $\tilde{J}(\tilde{v}_0, \tilde{K})$ it is reasonable to look for competitors that add some positivity set to the positivity set that v already has. This is the motivation for the following sections.

From now on we will drop the tildes and work exclusively in $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : -\cot(l/2)|x_1| < x_2\}$.

10.1 Reduction to a different optimization

To find a better competitor than $v = x_2^+$ we will construct a bounded set $E \subseteq \mathbb{R}^2_- = \{x_2 < 0\}$, with Lipschitz boundary, such that the following expression is arbitrarily small meanwhile keeping the size of $E \cap (\mathbb{R}^2 \setminus \Omega)$ not too small,

$$F(E) = |E| - \int_{\mathbb{R}} u_E(x_1, 0)dx_1,$$

where u_E is the solution of

$$\begin{aligned} \Delta u_E &= 0 \quad \text{in } E \cup \mathbb{R}^2_+, \\ u_E &= x_2^- \text{ in } \mathbb{R}^2 \setminus (E \cup \mathbb{R}^2_+), \end{aligned}$$

such that $u_E \to 0$ as $|x| \to \infty$.

Some properties of F are given by the following Lemma.

Lemma 10.1.1. *Given $E \subseteq \mathbb{R}^2_-$ bounded and with Lipschitz boundary, we have that the following hold,*

1. *Scaling: For $t > 0$ the scaled set tE satisfies $F(tE) = t^2 F(E)$.*

2. *Relation with J: For $u_{E,R}$ the solution of*

$$\begin{aligned} \Delta u_{E,R} &= 0 \quad \textit{in } E_R \cup B_R^+, \\ u_{E,R} &= x_2^- \textit{ in } \mathbb{R}^2 \setminus (E \cup B_R^+), \end{aligned}$$

Then for $v_{E,R} = u_{E,R} + x_2$,

$$F(E) = \lim_{R\to\infty} \left(J(v_{E,R}, B_R) - J(v, B_R)\right) \geq 0.$$

Proof. (1) follows by the change of variables formula because $u_{tE} = tu_E(t^{-1}\cdot)$.

To prove (2) we take first R sufficiently large such that $B_R \supseteq E$ and use that v minimizes $J(B_R)$ while $v_{E,R}$ is harmonic in $E \cup B_R^+$,

$$\begin{aligned} 0 &\leq J(v_{E,R}, B_R) - J(v, B_R), \\ &= |E| + \int_{E\cup B_R^+} |Dv_{E,R}|^2 - |Dv|^2, \\ &= |E| - \int_{E\cup B_R^+} |D(v_{E,R} - v)|^2. \end{aligned}$$

We use now that in $E \cup B_R^+$ the following holds in the distributional sense $\Delta(v_{E,R} - v) = -\Delta v = -\chi_{\{x_2=0\}}\mathcal{H}^1$.

$$\begin{aligned} 0 &\leq J(v_{E,R}, B_R) - J(v, B_R), \\ &= |E| - \int_{\mathbb{R}} (v_{E,R} - v)(x_1, 0)dx_1, \\ &= |E| - \int_{\mathbb{R}} u_{E,R}(x_1, 0)dx_1. \end{aligned}$$

Sending $R \to \infty$ makes $u_{E,R} \to u_E$ uniformly in the bounded set $\bar{E} \cap \{x_2 = 0\}$ and therefore also in $\{x_2 = 0\}$ because both functions are zero in $\{x_2 = 0\} \setminus (\bar{E} \cap \{x_2 = 0\})$. This implies that the integral of $u_{E,R}(\cdot, 0)$ converges to the integral of $u_E(\cdot, 0)$ and this concludes the Lemma. □

Remark 10.1.2. *The previous proof also works to show that,*

$$F(E) - |E \cap (\mathbb{R}^2 \setminus \Omega)| \geq \lim_{R\to\infty} \left(J(v_{E,R}, B_R \cap \Omega) - J(v, B_R \cap \Omega)\right).$$

We just have to notice that,

$$
\begin{aligned}
&J(v_{E,R}, B_R \cap \Omega) - J(v, B_R \cap \Omega) \\
&\leq |E| - |E \cap (\mathbb{R}^2 \setminus \Omega)| + \int_{E \cup B_R^+} |Dv_{E,R}|^2 - |Dv|^2.
\end{aligned}
$$

In this sense we can make clear what is our strategy. By finding E such that $F(E) - |E \cap (\mathbb{R}^2 \setminus \Omega)| < 0$ we would be able to get a better competitor than v in $B_R \cap \Omega$ for some R sufficiently large.

10.2 Initial step

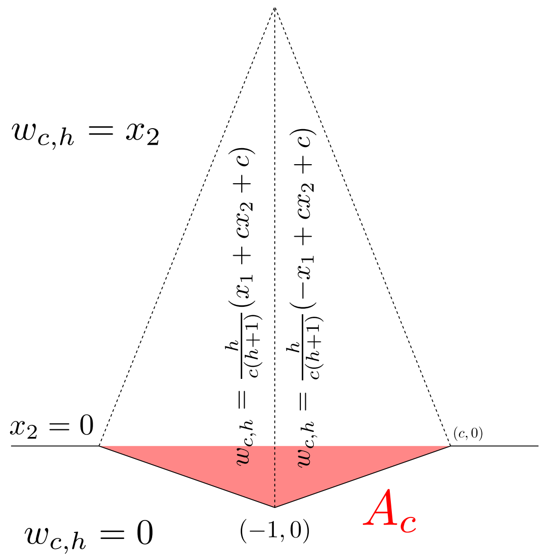
The following Lemma gives an estimate of F in isosceles triangles. This will be the basic configuration which we will use in our inductive construction.

Lemma 10.2.1. *Given $c > 0$, let A_c the isosceles triangle with vertices $(-c, 0), (c, 0)$ and $(-1, 0)$. Then $F(A_c) \leq 2$.*

Proof. Let $B_{c,h}$ be the quadrileteral with vertices at $(-c, 0), (0, h), (c, 0)$ and $(-1, 0)$. We construct first a function $w_{c,h}$ such that $J(w_{c,h}, B_{c,h}) - J(v, B_{c,h}) \leq 2$. Let for $(x_1, x_2) \in B_{c,h} \cap \{x_1 \leq 0\}$,

$$
w_{c,h}(x_1, x_2) = \frac{h}{c(h+1)}(x_1 + cx_2 + c).
$$

For $(x_1, x_2) \in B_{c,h} \cap \{x_1 \geq 0\}$ we define $w_{c,h}$ by extending it symmetrically, $w_{c,h}(x_1, x_2) = w_{c,h}(-x_1, x_2)$. Outside of $B_{c,h}$ we just make $w_{c,h} = v$. Notice that $w_{c,h}$ is continuous across $\partial B_{c,h}$ and it is an admissible competitor against v in any ball $B_R \supseteq B_{c,h}$.



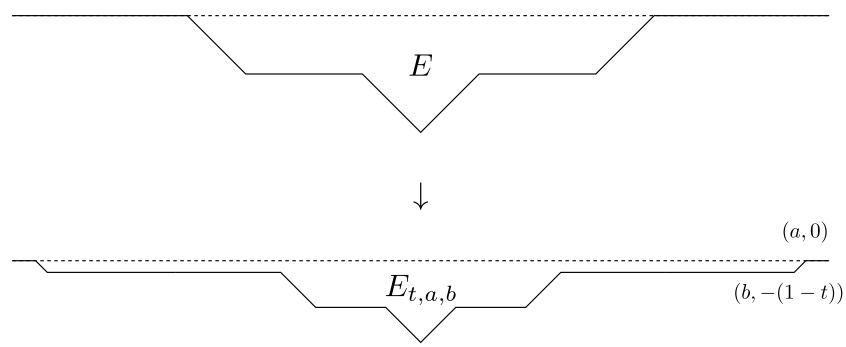
Let's compute the difference of the energies and then fix h so that it minimizes it,

$$J(w_{c,h}, B_R) - J(v, B_R) = c + \left(\frac{h^2}{h+1}\right)\left(\frac{c^2+1}{c}\right) - ch.$$

In order to minimize the previous expresion we chose $h = \sqrt{c^2+1} - 1$. The previous difference is now,

$$J(w_{c,h}, B_R) - J(v, B_R) = 2\frac{\sqrt{c^2+1} - 1}{c} \leq 2.$$

Now we replace $w_{c,h}$ by the harmonic function $v_{A_c,R}$ in $A_c \cup B_R^+$ taking the boundary values $v_R = w_{c,h} = v$ in $\partial(A_c \cup B_R^+)$. This makes $J(v_{A_c,R}, B_R) \leq J(w_{c,h}, B_R)$ and $J(v_{A_c,R}, B_R) - J(v, B_R) \leq 2$. By taking $R \to \infty$ and using Lemma 10.1.1 we obtain desired estimate for $F(A_c)$. $\square$



10.3 Inductive step

Now we describe how to diminish the value of $F(E)$ inductively meanwhile keeping $|E\cap(\mathbb{R}^2\backslash\Omega)|$ bounded away from zero. Consider a set $E \subset\subset \mathbb{R}\times[-1,0]$ and scale it by a factor $t \in (0,1)$, this diminishes the value of F by a factor t^2. The next step is to translate $tE \cup \bar{\mathbb{R}}^2_+$ downwards a distance $(1-t)$ giving us,

$$E_t = \Big(\big(tE \cup \bar{\mathbb{R}}^2_+\big) - (1-t)e_2\Big) \cap \mathbb{R}^2_- \subseteq \mathbb{R} \times [-1,0].$$

This set however is unbounded, so we truncate it by the trapezoid $T_{t,a,b}$, for $a > b > 0$, with vertices at $(-a,0),(a,0),(-b,-(1-t))$ and $(b,-(1-t))$, obtaining in this way,

$$E_{t,a,b} = E_t \cap T_{t,a,b} \subset\subset \mathbb{R} \times [-1,0].$$

Formally we expect $F(E_t)$ to be $t^2F(E)$ however here we are actually subtracting two infinite quantities. The intuition behinds this is that the downwards translation of tE adds as much volume as the amount in which the integral increases. We will see then that the truncation given by $T_{t,a,b}$ can be made such that it does not add to much to the functional. This is the motivation for the following Lemma.

Lemma 10.3.1. *Given $E \subset\subset \mathbb{R} \times [-1,0]$ with Lipschitz boundary and symmetric with respect to $\{x_1 = 0\}$ and $t \in (0,1)$ there exists $a_0 > b_0 > 0$ sufficiently large such that $F(E_{t,a,b}) \leq t^2 F(E) + 3(1-t)^2$ for any $a > \min(a_0, b)$ and $b > b_0$.*

Remark 10.3.2. *In the previous Lemma the optimal choice of t in order to minimize the upper bound for $F(E_{t,a,b})$ is*

$$t = \frac{3}{3 + F(E)}$$

for which

$$F(E_{t,a,b}) \leq \frac{3F(E)}{3 + F(E)}.$$

Proof. We rewrite $F(E_{t,a,b})$ in the following way,

$$\begin{aligned} F(E_{t,a,b}) &= |tE| + (a+b)(1-t) - \int_{\mathbb{R}} u_{E_{t,a,b}}(x_1, 0)dx_1, \\ &= |tE| - \int_{-b}^{b} \big(u_{E_{t,a,b}}(x_1,0) - (1-t)\big)\, dx_1, \\ &+ \left((a-b)(1-t) - 2\int_b^a u_{E_{t,a,b}}(x_1,0)dx_1 \right). \end{aligned}$$

Now we compare $(u_{E_{t,a,b}} - (1-t))$ with $\tilde{u}_t = u_{tE}(\cdot + (1-t)e_2)$ in order to include $F(tE) = t^2 F(E)$ in the right hand side. $\tilde{u}_t$ satisfies,

$$\begin{aligned} \Delta \tilde{u}_t &= 0 && \text{in } \left(tE \cup \bar{\mathbb{R}}^2_+\right) - (1-t)e_2, \\ \tilde{u}_t &= x_2^- - (1-t) && \text{in } \mathbb{R}^2 \setminus \left(\left(tE \cup \bar{\mathbb{R}}^2_+\right) - (1-t)e_2\right). \end{aligned}$$

Similarly $(u_{E_{t,a,b}} - (1-t))$ satisfies,

$$\begin{aligned} \Delta(u_{E_{t,a,b}} - (1-t)) &= 0 && \text{in } E_{t,a,b} \cup \bar{\mathbb{R}}^2_+, \\ (u_{E_{t,a,b}} - (1-t)) &= x_2^- - (1-t) && \text{in } \mathbb{R}^2 \setminus \left(E_{t,a,b} \cup \bar{\mathbb{R}}^2_+\right). \end{aligned}$$

Notice that sending $a \to \infty$ makes the domain $E_{t,a,b} \cup \bar{\mathbb{R}}^2_+$ to approach the domain $\left(tE \cup \bar{\mathbb{R}}^2_+\right) - (1-t)e_2$ locally with respect to the Hausdorff distance. This implies that as $a \to \infty$ we have that $(u_{E_{t,a,b}} - (1-t)) \to \tilde{u}_t$ locally uniformly. Given $\varepsilon > 0$, there is some a sufficiently large such that,

$$\begin{aligned} \int_{-b}^{b} \left(u_{E_{t,a,b}}(x_1,0) - (1-t)\right) dx_1 &\leq \int_{-b}^{b} \tilde{u}_t(x_1,0)dx_1 + \varepsilon, \\ &= \int_{-b}^{b} u_{tE}(x_1, 1-t)dx_1 + \varepsilon, \end{aligned}$$

We can then chose b sufficiently large such that, by using the Poison kernel of the half plane,

$$\begin{aligned} \int_{-b}^{b} \left(u_{E_{t,a,b}}(x_1,0) - (1-t)\right) dx_1 &\leq \int_{\mathbb{R}} u_{tE}(x_1, 1-t)dx_1 + 2\varepsilon, \\ &= \int_{\mathbb{R}} u_{tE}(x_1,0)dx_1 + 2\varepsilon. \end{aligned}$$

Giving us the following comparison between $F(E_{t,a,b})$ and $F(tE)$ for a and b sufficiently large,

$$F(E_{t,a,b}) \leq F(tE) + 2\varepsilon + \left((a-b)(1-t) - 2\int_{b}^{a} u_{E_{t,a,b}}(x_1,0)dx_1\right).$$

We will se now that the last term is controlled by $2(1-t)^2$. Then we will set $2\varepsilon = (1-t)^2$ to conclude the Lemma.

Let $c = (1-t)^{-1}(a-b)$ and $u_{(1-t)A_c}$ where the triangle A_c is the same from Lemma 10.2.1. We have the inclusion $(1-t)A_c + be_1 \subseteq E_{t,a,b}$ which implies that $u_{(1-t)A_c}(\cdot - be_1) \leq u_{E_{t,a,b}}$ and then,

$$\begin{aligned}
2\int_b^a u_{E_{t,a,b}}(x_1,0)dx_1 &\geq 2\int_b^a u_{(1-t)A_c}(x_1-b,0)dx_1,\\
&= \int_{\mathbb{R}} u_{(1-t)A_c}(x_1,0)dx_1,\\
&= |(1-t)A_c| - F((1-t)A_c),\\
&= (1-t)(a-b) - 2(1-t)^2.
\end{aligned}$$

Therefore,

$$(a-b)(1-t) - 2\int_b^a u_{E_{t,a,b}}(x_1,0)dx_1 \leq 2(1-t)^2.$$

Which is what we were looking for. $\square$

10.4 Proof of Theorem 10.0.7

The following Lemma combined with the previous Lemma 10.0.8 will complete the proof of Theorem 10.0.7.

Lemma 10.4.1. *Given $l < 2\pi$, there exits a set E and a radius R sufficiently large such that $J(v_{E,R}, B_R \cap \Omega) < J(v, B_R \cap \Omega)$.*

Proof. Let A_c the isocales triangle described in Lemma 10.2.1, let $D = A_c \cap (\mathbb{R}^2 \setminus \Omega) \cap \{x_2 \geq -3/5\}$ and try to find E such that:

1. $D \subseteq E$,

2. $F(E) < |D|$.

By having this we use the Remark 10.1.2 which says that

$$0 > F(E) - |E \cap (\mathbb{R}^2 \setminus \Omega)| \geq \lim_{R \to \infty} (J(v_{E,R}, B_R \cap \Omega) - J(v, B_R \cap \Omega))$$

and implies the Lemma.

Let $E_0 = A_c$, we know that,

1. $D \subseteq E_0$

2. $F(E_0) \leq 2$ from Lemma 10.2.1.

Given E_k let,

$$\begin{aligned} F_k &= F(E_k), \\ t_k &= \frac{3}{3 + F_k}, \\ E_{k+1} &= (E_k)_{t_k, a_k, b_k}. \end{aligned}$$

with a_k and b_k sufficiently large such that Lemma 10.3.1 applies and

$$F_{k+1} \leq \frac{3F_k}{3 + F_k}.$$

It is easy to show that such recurrence relation makes $F_k \to 0$ as $k \to \infty$. Eventually there will be some k_0 sufficiently large such that $F_{k_0} \leq |D|$. We now note that $F_{k_0} \leq |D|$ independently of how large c was chosen in constructing A_c. k_0 will only depend on the length l of the cone. Since we need to apply the iteration only k_0 times, we may choose c large enough in the construction of $A_c = E_0$ so that

$$D \subset E_{k_0}$$

Then we just have to chose $E = E_{k_0}$ to conclude the Lemma. $\square$

10.5 Stability

When we combine Theorem 10.0.7 with the stability given by Theorem 3.1.1 we are able to say that the vertex not only is not in the free boundary but stays away from it a given distance.

Corollary 10.5.1. *Let $l < 2\pi$ and u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ then there exists some $\varepsilon = \varepsilon(l) > 0$ such that $\mathcal{C}_\varepsilon \cap (\partial\Omega^+ \cup \partial\Omega^-) = \emptyset$.*

Proof. Proceed by contradiction assuming that there exists a sequence of minimizers $\{u_k\}_{k\in\mathbb{N}}$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ such that for $\Omega^+ = \{u_k > 0\} \cap \mathcal{C}_1$ and $\Omega^+ = \{u_k < 0\} \cap \mathcal{C}_1$ we have that

$$\mathcal{C}_{1/k} \cap (\partial\Omega_k^+ \cup \partial\Omega_k^-) \neq \emptyset.$$

By Theorem 9.3.2 the sequence is equicontinuous and also bounded therefore by Arzela-Ascoli it has a subsequence which converges uniformly to some function u_0 such that $0 \in (\partial\Omega_0^+ \cup \partial\Omega_0^-)$, with $\Omega^\pm$ defined in a similar way. By the stability given 3.1.1 we know that u_0 is a minimizer too but this contradicts Theorem 10.0.7. $\square$

Chapter 11

The vertex and the free boundary: Case $l \geq 2\pi$

In this chapter we discuss the problem of determining whether the vertex may belong to the free boundary in the case $l \geq 2\pi$. We show some examples when $0 \in (\partial\Omega^+ \cup \partial\Omega^-)$ using the well known fact that when $l = 2\pi$ and $\lambda_+ = 1, \lambda_- = 0$ then $u = x_2^+$ is a minimizer of this type. Moreover it is the unique minimizer of $J(K)$ for any compact set $K \subseteq \mathbb{R}^2$ subject to its own boundary values.

11.1 One phase free boundary through the vertex

Consider $l \geq 2\pi$, $\lambda_+ = 1, \lambda_- = 0$. The function $u = x_2^+$ defined in $\mathbb{R}^2 \setminus \{x_1 = 0, x_2 < 0\}$ can also be considered in $\mathcal{C}$ by using an isometry $\phi : \mathbb{R}^2 \setminus \{x_1 = 0, x_2 < 0\} \to U \subseteq \mathcal{C}$. Even though ϕ is not an isometry between $\mathbb{R}^2 \setminus \{x_1 = 0, x_2 < 0\}$ and $\mathcal{C}_1$, we can consider $\tilde{u} = u \circ \phi : U \to \mathbb{R}$ and then extend it to $\mathcal{C}$ by making it zero in $\mathcal{C} \setminus U$. We will drop now the tilde and consider $u = x_2^+$ defined in $\mathcal{C}$.

Let v be a function on $\mathcal{C}_1$ such that it has the same boundary values as u in $\mathcal{C}_1$ and minimizes $J(\mathcal{C}_1)$. We will show that $v \equiv u$. In the quotient $\mathcal{R}^2/\{\theta \in l\mathbb{Z}\}$ and after an appropriated rotation u can be considered as $u(r, \theta) = r \cos\theta$. It satisfies that $u(r,\theta) = u(r,-\theta)$. Let now $\tilde{v}$ be defined by $\tilde{v}(r,\theta) = v(r,-\theta)$. Both functions v and $\tilde{v}$ have the same boundary values a u in $\mathcal{C}_1$ and also $\tilde{v}$ minimizes $J(\mathcal{C}_1)$. Consider now $v^+ = \max(v,\tilde{v})$ and $v^- = \min(v,\tilde{v})$. By the lattice principle Lemma 12.3.1 both $v^\pm$ are minimizers of $J(\mathcal{C}_1)$ with the same boundary data and symmetry as u.

At this point we see that $v^\pm \circ \phi^{-1}$ also minimizes $J(B_1)$ with the same boundary values as $u = x_2^+$. The symmetry across $\{x_1 = 0\}$ implies that the Dirichlet term does not add to the functional if we include the segment $\{x_1 = 0, x_2 \in (0,-1)\}$. However $u = x_2^+$ was the unique minimizer to that problem and therefore $v^\pm \equiv u$, so $v \equiv u$. Going back to $\mathcal{C}$ we have found a minimizer with $0 \in (\partial\Omega^+ \cup \partial\Omega^-)$.

11.2 More than one positive phase free boundary through the vertex

The previous idea can be extended to construct examples where two positive phases meet at the vertex. This is something unexpected since in the case when $l = 2\pi$ we know that the free boundary is smooth.

Consider $l \geq 4\pi$, $\lambda_+ = 1, \lambda_- = 0$, $\mathcal{C}$ parametrized by $\mathcal{R}^2/\{\theta \in l\mathbb{Z}\}$ and two isometries,

$$\phi_+ : U^+ = \{\theta \in (-\pi,\pi)\} \to \mathbb{R}^2 \setminus \{x_2 = 0, x_1 < 0\},$$
$$\phi_- : U^- = \{\theta \in (l/2-\pi, l/2+\pi)\} \to \mathbb{R}^2 \setminus \{x_2 = 0, x_1 > 0\}$$

In this case the two functions $u_\pm = x_1^\pm$ can be pasted together to construct

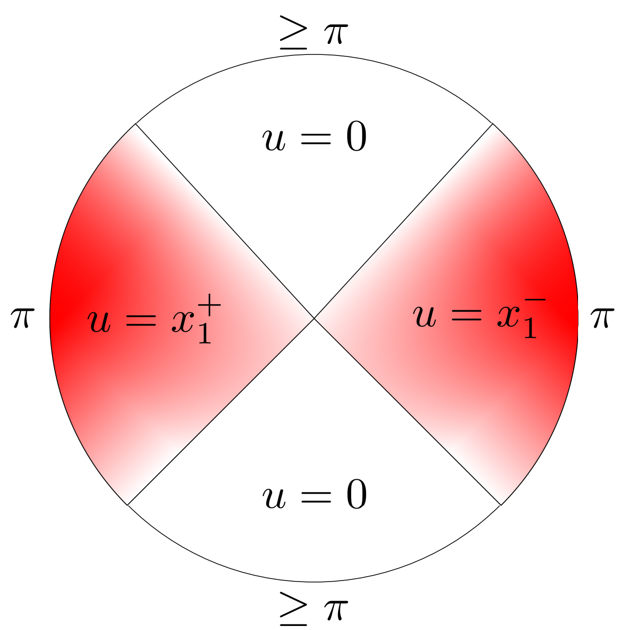


Figure 11.1: Pasting two linear pieces

a function $u = u^+ \circ \phi_+ + u^- \circ \phi_-$ such that $u = u^\pm \circ \phi_\pm$ in $U^\pm$. We now consider a competitor v. If v is a minimizer, we may use the lattice principle as before so that we may assume symmetry for v across the lines that would be horizontal and vertical in Figure 11.1. By cutting along the vertical line, we may use each half of v as a competitor against x_1^+ on the cone $\tilde{\mathcal{C}}_1$ which has half the lenth of the cone $\mathcal{C}_1$. If $J(v) \leq J(u)$ on $\mathcal{C}$, then necessarily each half must minimize, so $J(v) \leq J(x_1^+)$ on $\tilde{\mathcal{C}}_1$. As shown in Section 11.1 above, x_1^+ is the unique minimizer subject to its own boundary values on $\tilde{\mathcal{C}}_1$, so we conclude each half of v is identical to x_1^+.

This construction can also be generalized to show that k phases can meet at the vertex if $l \geq 4k\pi$.

Chapter 12

Appendix

12.1 Monotonicity formulas

Monotonicity formulas for harmonic and subharmonic functions allow us to control infinitesimal quantities by integral ones. The classical monotonicity for the average of the Dirichlet energy of a harmonic function or the Alt-Caffarelli-Friedman (ACF) formula can be applied when the domain of integration doesn't contain the vertex. When we decide to center the integrals at the vertex then they are no longer valid and the classical proofs have to be slightly modified.

Given $r > 0$, we fix $\mathcal{C}_r$ to be the intersection of $\mathcal{C}$ with the ball of radius r centered at the origin.

Lemma 12.1.1 (Monotonicity of the average Dirichlet energy)**.** *Let u be a harmonic function over the cone $\mathcal{C}$ with length l. Then*

$$\frac{1}{r^{2\alpha}} D(\mathcal{C}_r, u) = \frac{1}{r^{2\alpha}} \int_{\mathcal{C}_r} |Du|^2$$

is an increasing function of r for $\alpha \in (0, 2\pi/l]$

Proof. Integrating by parts,

$$\frac{1}{r^{2\alpha}}D(\mathcal{C}_r) = \frac{1}{r^{2\alpha}}\int_{\partial\mathcal{C}_r} uu_r d(r\theta).$$

Now we use the Fourier representation of u and the fact that the sequence of functions given by the sines and cosines are and orthogonal set in $L^2(\mathcal{C}_r)$. Let

$$u = \sum_{k=0}^{\infty} r^{2\pi k/l}\left(a_k \cos\frac{2\pi k}{l}\theta \;+\; b_k \sin\frac{2\pi k}{l}\theta\right),$$

then

$$\begin{aligned}\frac{1}{r^{2\alpha}}\int_{\partial\mathcal{C}_r} uu_r &= \sum_{k=0}^{\infty}(2\pi k/l)r^{4\pi k/l-2\alpha}\int_0^l a_k^2\cos^2\frac{2\pi k}{l}\theta + b_k^2\sin^2\frac{2\pi k}{l}\theta d\theta,\\ &= \pi\sum_{k=1}^{\infty} kr^{4\pi k/l-2\alpha}(a_k^2+b_k^2).\end{aligned}$$

As $\alpha \in (0, 2\pi/l]$, the exponents appearing on the sum above are all non negative, each term the is non decreasing in r and the whole series is therefore non decreasing in r. $\square$

Remark 12.1.2. *At any other point $x_0 \neq 0$ we can also define the ball $B_r(x_0) \subseteq \mathcal{R}^2$. As far as $r \leq |x_0|$ this ball looks exactly as the flat ball we are use to. In that case the monotonicity proof given above works with any exponent $2\alpha \in (0, 1]$. Therefore we obtain for $\alpha \in (0, \min(1, 2\pi/l)]$,*

$$\frac{1}{r^{\alpha}}D(B_r(x_0), u) = \frac{1}{r^{2\alpha}}\int_{B_r(x_0)} |Du|^2,$$

is also increasing with the restriction that $r \leq |x_0|$ if $x_0 \neq 0$.

Lemma 12.1.3 (Alt-Caffarelli-Friedman monotonicity formula)**.** *Let $\{u_+, u_-\}$ be a pair of nonnegative continuous subharmonic functions on the cone $\mathcal{C}_1$ with length $l \leq 2\pi$ such that $u_+ \cdot u_- = 0$ in $\mathcal{C}_1$ and $\alpha \in (0, 4\pi/l]$. Then the functional*

$$r \mapsto \Phi(r, u_+, u_-) = \frac{1}{r^{2\alpha}} \int_{\mathcal{C}_r} |Du_+|^2 \int_{\mathcal{C}_r} |Du_-|^2$$

is nondecreasing for $0 < r < 1$.

Proof. As in the classical proof we have that,

$$\begin{aligned} \frac{r\Phi'(r)}{2\Phi(r)} &= -\alpha + \frac{\int_{\partial \mathcal{C}_1} |Du^+|^2}{2\int_{C_1} |Du^+|^2} + \frac{\int_{\partial C_1} |Du^-|^2}{2\int_{C_1} |Du^-|^2}, \\ &\geq -\alpha + \left(\frac{\int_0^l (\tilde{u}_\theta^+)^2 d\theta}{\int_0^l (\tilde{u}^+)^2 d\theta} \right)^{1/2} + \left(\frac{\int_0^l (\tilde{u}_\theta^+)^2 d\theta}{\int_0^l (\tilde{u}^-)^2 d\theta} \right)^{1/2}. \end{aligned}$$

The last two terms get minimized by the first eigenvalues of the support of $u^\pm$. They become even smaller if we assume that each one of these two domains are connected and have complementary lengths m and $l - m$. In that case the eigenvalues are $-(\pi/m)^2$ and $-(\pi/(l-m))^2$. So the expression above gets minimized when $m = l/2$ and then,

$$\frac{r\Phi'(r)}{2\Phi(r)} \geq -\alpha + 4\pi/l,$$

which is non negative for $\alpha \in (0, 4\pi/l]$. □

12.2 Initial regularity

We can get some regularity for the minimizer u by just comparing it with its harmonic replacement in a given ball.

Lemma 12.2.1. *Let u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$, then*

$$|u(x) - u(0)| \leq C|x|^{\alpha/3} \text{ for every } x \in \mathcal{C}_{1/10},$$

for any $\alpha \in (0, \min(1, 2\pi/l))$ and some universal $C = C(\alpha) > 0$.

Proof. We prove first that for every $x_0 \in B_{1/2}$, $r \in (0, 1/2)$ and $\alpha \in (0, \min(1, 2\pi/l))$,

$$\int_{\mathcal{C}_r} |Du|^2 \leq Cr^{2\alpha}. \tag{12.2.1}$$

Consider $0 < r < R < 1/2$ and h_R the harmonic function in $\mathcal{C}_R$ taking the same boundary values as u in $\partial\mathcal{C}_R$. Then v is an admissible competitor for J against u in $\mathcal{C}_R$ from where we get,

$$\int_{\mathcal{C}_R} |Du|^2 - |Dh_R|^2 \leq CR^2.$$

Because h_R is harmonic and $u - h_R \in H^1_0(\mathcal{C}_R)$,

$$\int_{\mathcal{C}_R} |D(u - h_R)|^2 = \int_{\mathcal{C}_R} |Du|^2 - |Dh_R|^2 \leq CR^2.$$

Now we estimate how much $D(u)$ grows from r to R from Lemma 12.1.1 and the fact that h_R minimizes the Dirichlet energy in $\mathcal{C}_R$,

$$\begin{aligned}
\int_{\mathcal{C}_r} |Du|^2 &\leq \int_{\mathcal{C}_R} |D(u - h_R)|^2 + \int_{\mathcal{C}_r} |Dh_R|^2, \\
&\leq CR^2 + \left(\frac{R}{r}\right)^{4\pi/l} \int_{\mathcal{C}_R} |Dh_R|^2, \\
&\leq CR^2 + \left(\frac{R}{r}\right)^{4\pi/l} \int_{\mathcal{C}_R} |Du|^2.
\end{aligned}$$

From this we conclude (12.2.1) by applying Lemma 3.4 in [28].

Now we proof, in a similar way as in the Morrey estimates, that for $R \in (0,1/2)$,

$$\int_{\mathcal{C}_R} \frac{|u(y)-u(0)|}{|y|} dy \leq CR^{1+\alpha}. \tag{12.2.2}$$

The following computations can be made rigorous after regularizing u by a convolution. We obtain the slope of u between 0 and y by performing the following integral,

$$\frac{|u(y)-u(0)|}{|y|} \leq \int_0^1 |Du(ty)|dt.$$

Next we integrate in $\partial\mathcal{C}_r$,

$$\int_{\partial\mathcal{C}_r} \frac{|u(y)-u(0)|}{|y|} \leq \int_0^1 \frac{dt}{t} \int_{\partial\mathcal{C}_{tr}} |Du|.$$

Finally we integrate with respect to r between 0 and R, apply Hölder's inequality and the previous estimate (12.2.1),

$$\begin{aligned}
\int_{\mathcal{C}_R} \frac{|u(y)-u(0)|}{|y|} &\leq \int_0^1 R^2 dt \frac{1}{(tR)^2} \int_{\mathcal{C}_{tR}} |Du|,\\
&\leq \int_0^1 R^2 dt \left(\frac{1}{(tR)^2} \int_{\mathcal{C}_{tR}} |Du|^2 \right)^{1/2},\\
&\leq CR^{1+\alpha} \int_0^1 t^{-1+\alpha} dt.
\end{aligned}$$

As $\alpha > 0$ the integral above is finite and we conclude (12.2.2).

Finally we use (12.2.2) to compare $u(0)$ with $u(x)$ with $x \in \mathcal{C}_{1/10}$. Consider a parameter $\varepsilon \in (0,1/2)$ to be fixed and $r = \varepsilon|x|$, we apply first the triangular inequality and then integrate over $B_{\varepsilon r}(x)$ using Proposition 9.3.1,

$$\begin{aligned} |u(x)-u(0)||B_{\varepsilon r}(x)| &\leq \int_{B_{\varepsilon r}(x)} |u(y)-u(x)|dy + \int_{\mathcal{C}_{2r}} |u(y)-u(0)|dy, \\ &\leq Cr^{-1}\int_{B_{\varepsilon r}(x)} |x-y|dy + 2r\int_{\mathcal{C}_{2r}} \frac{|u(y)-u(0)|}{|y|}dy, \\ &\leq C\left(\varepsilon^3 r^2 + r^{2+\alpha}\right), \end{aligned}$$

It implies that $|u(x)-u(0)| \leq C(\varepsilon + r^\alpha\varepsilon^{-2})$, then we just chose $\varepsilon = r^{\alpha/3}(\leq 10^{-1/3} < 1/2)$ to conclude the proof. $\square$

Here is the proof of Theorem 9.3.2. Notice that the estimate degenerates in two ways, as l grows and also as the Hölder exponent goes to $\min(1, 2\pi/l)$.

Proof of Theorem 9.3.2. Let $x, y \in \mathcal{C}_{1/10}$ and assume without lost of generality that y is closest one to 0. Let $\varepsilon \in (0, 1/2)$ a parameter to be fixed and $r = |x|$. We consider two cases according if y belongs or not to $B_{\varepsilon r}(x)$.

If $y \in B_{\varepsilon r}(x)$. Then we use the Lipschitz estimate from Proposition 9.3.1 to get

$$|u(x)-u(y)| \leq Cr^{-1}|x-y|.$$

Given that $\varepsilon \leq r^{\alpha/(4-\alpha)}(\leq 10^{-1/3} < 1/2)$ we obtain that $r^{-1}|x-y| \leq |x-y|^{\alpha/4}$.

If $y \notin B_{\varepsilon r}(x)$ then we use the previous Lemma to get that

$$\begin{aligned} |u(x)-u(y)| &\leq |u(x)-u(0)| + |u(y)-u(0)|, \\ &\leq Cr^{\alpha/3}. \end{aligned}$$

Given that $\varepsilon = r^{1/3}(\leq r^{\alpha/(4-\alpha)})$ we obtain $r^{\alpha/3} \leq (\varepsilon r)^{\alpha/4} \leq |x-y|^{\alpha/4}$. $\square$

12.3 Stability

We start by proving a non degeneracy estimate at the vertex. As we have done before we will use the already known results for the flat metric case when the problem is considered away from the origin. At the origin we will a non degeneracy result. The following Lattice Principle will be used to obtain non degeneracy.

Lemma 12.3.1 (Lattice Principle)**.** *Let u, v be two minimizers on $\mathcal{C}_r$ with $u \leq v$ on $\partial\mathcal{C}_r$. Then $\underline{w} = \min\{u, v\}$ and $\overline{w} = \max\{u, v\}$ are minimizers on $\mathcal{C}_r$ subject to their respective boundary conditions.*

Proof. One may easily check that

$$J(\underline{w}) + J(\overline{w}) = J(u) + J(v)$$

Since $\underline{w} = u$ and $\overline{w} = v$ on $\partial\mathcal{C}_r$ it follows that $\underline{w}$ and $\overline{w}$ are minimizers of J. □

Lemma 12.3.2 (Non degeneracy at the vertex)**.** *Let u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with λ_+ and λ_- different from zero. For $\varepsilon \in (0, 1)$ there exists some $\delta > 0$ such that $|u| < \delta$ in $\mathcal{C}_\varepsilon$ implies $u = 0$ in $\mathcal{C}_{\varepsilon/2}$.*

Proof. Given $\delta > 0$ we will consider a competitor φ_δ which minimizes J with constant boundary value δ in $\partial\mathcal{C}_\varepsilon$. By Lemma 12.3.1 and Theorem 3.1.1 we may take the sup of all minimizers and conclude there is a unique minimizer φ_δ that lies above every other minimizer with constant boundary value δ. Any rotation of φ_δ is again a minimizer, and so φ_δ is a radially symmetric minimizer φ_δ. Given that δ is sufficiently small one may easily compute that

$$\varphi_\delta(x) = \lambda_+ r_0 \ln^+(r_0/|x|),$$

where r_0 is the largest of the two roots of $\lambda_+ r_0 \ln(r_0/\varepsilon) = \delta$. In particular φ_δ vanishes in $B_{\varepsilon/2}$ if we chose δ small enough.

Assuming that $|u| < \delta$ by Lemma 12.3.1 we have that $v = \max(\varphi_\delta, u)$ is a minimizer over $\mathcal{C}_\varepsilon$ and as stated above $v = \max(\varphi_\delta, u) \leq \varphi_\delta$. Then u vanishes in $B_{\varepsilon/2}$. $\square$

Corollary 12.3.3 (Stability of the zero set). *Let u_1 and u_2 be minimizers of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with λ_+ and λ_- different from zero. For any $\varepsilon \in (0, 1/2)$ there exists some $\delta > 0$ such that $|u_1 - u_2| < \delta$ implies $\{u_1 = 0\} \cap \mathcal{C}_1$ and $\{u_2 = 0\} \cap \mathcal{C}_1$ are ε-close in the Hausdorff distance.*

Proof. We have to show that $\{u_1 = 0\} \cap \mathcal{C}_1 \subseteq (\{u_2 = 0\} \cap \mathcal{C}_1) \oplus B_\varepsilon$ and by interchanging the roles of u_1 and u_2 we would have concluded the corollary. If the vertex doesn't belong to $\{u_1 = 0\} \cap \mathcal{C}_1$ then the result follows from the classical theory by isolating the vertex. So we will assume in this proof that $u_1(0) = 0$. The idea is to use the compactness of $\{u_1 = 0\} \cap \bar{\mathcal{C}}_1$ to put togheter the results away from the origin and at the origin.

For $x \in \{u_1 = 0\} \cap (\mathcal{C}_1 \setminus \{0\})$ we can use the classical theory in a ball $B_{r(x)}(x)$ with $r(x) = \min(|x|, \varepsilon/2)$ to conclude that there is some $\delta(x) > 0$

such that if $|u_1 - u_2| < \delta(x)$ in $B_{r(x)}(x)$, then $\{u_2 = 0\} \cap B_{r(x)}(x) \neq \emptyset$. Notice however that $\delta(x)$ degenerates as $x \to 0$.

We use the previous Lemma in the vertex in following form. Assume with out lost of generality that $u_2(0) \in (0, \delta_0)$ and lets see that $\{u_2 = 0\} \cap \mathcal{C}_{\varepsilon/2} \neq \emptyset$ if we chose δ_0 sufficiently small. Assume by contradiction that u_2 is a harmonic positive function in $\mathcal{C}_{\varepsilon/2}$. By Harnack's inequality $u_2 \in (0, C\delta_0)$ in $\mathcal{C}_{\varepsilon/4}$ and by having that δ_0 is small enough we obtain a contradiction with the previous Lemma.

Consider the covering of $\{u_1 = 0\} \cap \bar{\mathcal{C}}_1$ given by $\{B_{r(x)}(x)\} \cup \mathcal{C}_{\varepsilon/2}$ for x ranging over $\{u_1 = 0\} \cap (\bar{\mathcal{C}}_1 \setminus \{0\})$. Extract then a finite collection $x_1, \ldots, x_N$ such that $\mathcal{C}_{\varepsilon/2}, B_{r(x_1)}(x_1), \ldots, B_{r(x_N)}(x_N)$ still covers $\{u_1 = 0\} \cap \bar{\mathcal{C}}_1$ and chose δ to be the smallest number among $\delta_0, \delta(x_1), \ldots, \delta(x_N)$. From the previous considerations we have that $\{u_2 = 0\} \cap \mathcal{C}_1$ hits each one of the sets $\mathcal{C}_e, B_{r(x_1)}(x_1), \ldots, B_{r(x_N)}(x_N)$ which implies that for every $x \in \{u_1 = 0\} \cap \bar{\mathcal{C}}_1$ there is some $y \in \{u_2 = 0\} \cap \mathcal{C}_1$ such that $\operatorname{dist}(x, y) < \varepsilon$. This is equivalent to say that $\{u_1 = 0\} \cap \mathcal{C}_1 \subseteq \{u_2 = 0\} \cap \mathcal{C}_1 \oplus B_\varepsilon$ which concludes the proof. $\square$

Here is the proof of Theorem 3.1.1

Proof of Theorem 3.1.1. By the regularity already proved in Theorem 9.3.2 we have that the sequence is uniformly in $C^{\alpha/3}(\mathcal{C}_{1/2})$. By Arzela-Ascoli we have that the sequence has an accumulation point $v \in C^{\alpha/3}(\mathcal{C}_{1/2})$ with respect to the C^β normn for $\beta < \alpha/3$. By having that $u_k \to u$ in $L^2(\mathcal{C}_{1/2})$ we obtain that u is the only possible accumulation point in $L^2(\mathcal{C}_{1/2})$ and therefore $v = u$

and the whole sequence converges uniformly to u which proves the first part. The second part follows now from Corollary 12.3.3.

To conclude that u is a minimizer of J we use as in the classical proof the lower semicontinuity of the Dirichlet term and then the uniform convergence of $\{u_k > 0\} \cap \mathcal{C}_{1/2}$ and $\{u_k < 0\} \cap \mathcal{C}_{1/2}$ to $\{u > 0\} \cap \mathcal{C}_{1/2}$ and $\{u < 0\} \cap \mathcal{C}_{1/2}$ respectively. $\square$

12.4 Optimal regularity

The following Lemma and its Corollary gives a gradient bound at the free boundary points. Recall that for a set Ω we have defined $\Omega^+ = \Omega \cap \{u > 0\}$ and Ω^- is defined similarly.

Lemma 12.4.1. *Let $l \leq 2\pi$, u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ and let $x_0 \in (\partial\Omega^+ \cap \partial\Omega^-) \cap (\mathcal{C}_{1/2} \setminus \{0\})$; then $|Du(x_0)| \leq C$ for some universal constant $C > 0$.*

Proof. We use that $u \circ \phi_l^{-1} : \mathcal{R}^2 \to \mathbb{R}$ minimizes J over any compact set $\tilde{K} = \phi_l^{-1}(K)$ with $K \subseteq \mathcal{C}_1$ compact in order to know that u has enough regularity around x_0. The idea is that we apply the classical ACF monotonicity formula to $u^\pm$, centered at x_0 and, as the radius goes to zero, we measure the product of $|Du^\pm(x_0)|^2$.

$$2|Du^+(x_0)|^2|Du^-(x_0)|^2 \leq 4\frac{1}{|x_0|^4}\int_{B_{|x_0|(x_0)}} |Du^+|^2 \int_{B_{|x_0|(x_0)}} |Du^-|^2.$$

Now we apply the ACF monotonicity formula given by Lemma 12.1.3 with $\alpha = 4\pi/l \leq 2$. Notice that in order to apply such Lemma we are actually using Corollary 9.3.3.

$$
\begin{aligned}
|Du^+(x_0)|^2|Du^-(x_0)|^2 &\leq 64|x_0|^{2\alpha-4}\frac{1}{|x_0|^{2\alpha}}\int_{\mathcal{C}_{2|x_0|}}|Du^+|^2\int_{\mathcal{C}_{2|x_0|}}|Du^-|^2,\\
&\leq C\|Du^+\|^2_{L^2(\mathcal{C}_1)}\|Du^-\|^2_{L^2(\mathcal{C}_1)},\\
&\leq C.
\end{aligned}
$$

The minimizer u also satisfies the Euler Lagrange equation at x_0 in the classical sense, $|Du^+(x_0)|^2 - |Du^-(x_0)|^2 = \lambda_+^2 - \lambda_-^2 \neq 0$. Assume without lost of generality that $\lambda_+^2 - \lambda_-^2 = \Lambda > 0$. It implies

$$
\begin{aligned}
&|Du^-(x_0)|^2 \leq C\Lambda^{-1},\\
&|Du^+(x_0)|^2 = |Du^-(x_0)|^2 + \Lambda \leq C\Lambda^{-1} + \Lambda.
\end{aligned}
$$

□

Corollary 12.4.2. *Let $l \leq 2\pi$, u be a minimizer of $J = J(\mathcal{C}_1, \lambda_+, \lambda_-)$ with $\|Du\|_{L^2(\mathcal{C}_1)} \leq 1$ and let $x_0 \in (\partial\Omega^+ \cup \partial\Omega^-) \cap (\mathcal{C}_{1/2} \setminus \{0\})$; then $|Du(x_0)| \leq C$ for some universal constant $C > 0$.*

Proof. From the previous Lemma the only case left is when $x_0 \in (\partial\Omega^+\Delta\partial\Omega^-)\cap(\mathcal{C}_{1/2} \setminus \{0\})$. In such case u keeps just one sign in a neighbirhood of x_0 (either non negative or non positive) and minimizes a one phase problem in the same neighborhood. From the gradient bound for the flat case we obtain the gradient bound, independent of the distance to the vertex. □

We split the proof of Theorem 9.3.5 into two Lemmas depending if $0 \in \partial\mathcal{C}_1^{\pm}$ or not.

Lemma 12.4.3. *Let u and x_0 be as in Theorem 9.3.5 and assume additionally that $0 \in \mathcal{C}_1^+$. Then the same conclusions as in Theorem 9.3.5 hold.*

Proof. Let $d = \operatorname{dist}(0, \partial\mathcal{C}_1^+) \in (0, 1/2]$. The ball $\mathcal{C}_d$ touches $\partial\mathcal{C}_{1/2}^+$ at some point x_1 where we know that $|Du^+(x_1)| \leq C$ from Lemma 12.4.1. By using Harnack's inequality we get that $u(x) \geq Cu(0)$ in $B_{d/2}$ and then the following barrier can be put below u in $\mathcal{C}_d \setminus \mathcal{C}_{d/2}$ for c sufficiently small,

$$\varphi(x) = cu(0)(\ln|x_0| - \ln|x|).$$

This implies $C \geq |D\varphi(x_1)| = u(0)/d$. Which is the desired estimate at the origin.

For $x_0 \in \mathcal{C}_{d/2}$ we use Harnack's inequality to get that $u(x_0) \leq Cd \leq C \operatorname{dist}(x_0, \partial(\Omega^+ \cap \mathcal{C}_{1/2}))$. For x_0, now in $\mathcal{C}_{d/8} \setminus \{0\}$, we use the monotonicity of the Dirichlet energy,

$$\begin{aligned}
|Du(x_0)|^2 &\leq \frac{1}{|x_0|^2} \int_{B_{|x_0|}(x_0)} |Du|^2, \\
&\leq 4 \frac{1}{(2|x_0|)^2} \int_{\mathcal{C}_{2|x_0|}} |Du|^2, \\
&\leq C \frac{1}{d^2} \|Du\|^2_{L^2(\mathcal{C}_{d/4})}, \\
&\leq C \frac{1}{d^4} \|u\|^2_{L^2(\mathcal{C}_{d/2})}, \\
&\leq C
\end{aligned}$$

Which are the desired estimates at $\mathcal{C}_{d/8}$.

Finally we consider $x_0 \in \mathcal{C}_{1/4} \setminus \mathcal{C}_{d/8}$. Let $B_R(x_0)$ be the largest ball contained in $\mathcal{C}^+_{1/2} \setminus \{0\}$. If $B_R(x_0) \cap \partial\mathcal{C}^+_{1/2} \ni x_1$ then the estimates for x_0 follow by using Lemma 12.4.1 at x_1 and considering a lower barrier as before. If $R = |x_0|$ we also use a similar barrier and instead of Lemma 12.4.1 we use the estimates just proved at $\mathcal{C}_{d/2}$. Let,

$$\varphi(x) = cu(x_0)(\ln R - \ln |x|),$$

with c small enough such that by using Harnack's inequality we can get that $u \geq \varphi$ in $B_R(x_0) \setminus B_{R/2}(x_0)$. Because $u^+ \leq Cd$ in $\mathcal{C}_{d/2}$ we get that $C \geq |D\varphi(0)| \geq u(x_0)/R$. It implies that $u(x_0) \leq CR \leq C \operatorname{dist}(x_0, \partial(\Omega^+ \cap \mathcal{C}_{1/2}))$. For the gradient estimate we can use interior estimates at $B_R(x_0)$, i.e. $|Du(x_0)| \leq C|u(x_0)|/R \leq C$. $\square$

Lemma 12.4.4. *Let u and x_0 be as in Theorem 9.3.5 and assume additionally that $0 \in \partial\mathcal{C}^+_1$. Then the same conclusions as in Theorem 9.3.5 hold.*

Proof. The idea is to use a covering argument to pass the estimates from points that are close to $\partial\mathcal{C}^+_1$ to every other point in the positivity set.

Let $x_0 \in \mathcal{C}^+_{1/4} \setminus \{0\}$ and assume that for $r = |x_0|/2$, $B_r(x_0) \cap \{u^+ = 0\} \neq \emptyset$. Then the estimate follows as before by using Lemma 12.4.1 because for $d = \operatorname{dist}(x_0, \partial\mathcal{C}^+_{1/2})$ the ball $B_d(x_0)$ doesn't contain the vertex.

For general $x_0 \in \mathcal{C}^+_{1/4} \setminus \{0\}$ we consider a finite covering of $\partial\mathcal{C}_r$ with balls center at $\mathcal{C}_r$ and radius $r/2$ where $r = |x_0|$. Because $\{u = 0\} \cap \mathcal{C}_{1/2} \neq \emptyset$

there is one of these balls that intersects $\{u^+ = 0\}$ and then the estimates are valid there. To obtain the estimates at x_0 we just need to apply Harnack's inequality in a finite chain of balls up to one that reaches x_0.

To conclude let us notice that the gradient is not necessarily well defined at the vertex because for $l \neq 2\pi$ the tangent space at 0 is not well defined. Still the previous gradient estimate holds uniformly up to the vertex. $\square$

These two previous Lemmas conclude the proof of Theorem 9.3.5.

12.4.0.5 Blows-up

The first part in Corollary 9.3.6 follows from the previous stability and optimal regularity.

Proof of the first part in Corollary 9.3.6. Let $K \subseteq \mathcal{C}$ be a compact set. By the scaling of the functional we have that $\|u_k\|_{H^1(2K)}$ is uniformly bounded starting at some k_0 sufficiently large. There exists then an accumulation point $u \in H^1(2K)$ which is also a minimizer in K by Theorem 3.1.1. Moreover the whole sequence converges uniformly to u in K by the same Theorem. By the definition of the rescaling we have that the same sequence is uniformly bounded in $C^{0,1}(K)$ and therefore there is an acculumation point in $C^{0,1}(K)$. Because the sequence already converged to u uniformly we conclude that $u \in C^{0,1}(K)$ and the convergence happened also in $C^{0,\beta}(K)$ for $\beta < 1$. $\square$

For the second part in Corollary 9.3.6 we need to use a monotonicity formula as in [40]. There is also a similar monotonicity formula in [39]. The proofs of such monotonicity formulas use radial variations which naturally adapt to our situation with a cone. We reproduce the proof in [40] here. Notice also that the proof works no matter the length l of the cone, however for $l \geq 2\pi$ with $0 \in \partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$, since the optimal reguarity is unknown it might be possible for $W(\mathcal{C}_r, u) = -\infty$.

Theorem 12.4.5. *Let u be a minimizer of $J(\mathcal{C}_1, \lambda_+, \lambda_-)$ such that $u(0) = 0$ and define the* Weiss energy *for $r \in (0,1)$,*

$$\begin{aligned}
W(\mathcal{C}_r, u) &= \frac{1}{r^2} J(\mathcal{C}_r, u) - \frac{1}{r} \int_0^r \int_{\partial\mathcal{C}_1} \left(Du(tx) \cdot x\right)^2 dt, \\
&= \frac{1}{r^2} \left(\int_{\mathcal{C}_r} |Du|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}} \right), \\
&- \frac{1}{r} \int_0^r \int_{\partial\mathcal{C}_1} \left(Du(tx) \cdot x\right)^2 dt
\end{aligned}$$

Then $W(\mathcal{C}_r, u)$ is monotone increasing in r. Furthermore, if $0 < r_1 < r_2 < 1$, then $W(\mathcal{C}_{r_1}, u) = W(\mathcal{C}_{r_2}, u)$ if and only if u is homogeneous of degree 1 with respect to 0 on the ring $\mathcal{C}_{r_2} \setminus \mathcal{C}_{r_1}$.

Remark 12.4.6. *For $u_r = r^{-1}u(r\cdot)$, the functional W enjoys the following rescaling property:*

$$W(\mathcal{C}_R, u_r) = W(\mathcal{C}_{rR}, u_r)$$

Proof. An admissible competitor against u in $\mathcal{C}_t$ is given by the following 1-homogeneous function constructed from the trace of u in $\partial\mathcal{C}_t$. For $x \neq 0$ we denote $\vec{\theta} = x/|x|$,

$$u_t(x) = \frac{|x|}{t} u\left(t\vec{\theta}\right).$$

Moreover by Proposition 9.3.1 we have that u is Lipschitz in $\bar{\mathcal{C}}_t$ so that the following computation is well justified.

$$\begin{aligned} &\int_{\mathcal{C}_r} |Du_t|^2 + \lambda_+ \chi_{\{u_t>0\}} + \lambda_- \chi_{\{u_t,0\}}, \\ &= \frac{t}{2} \int_{\partial\mathcal{C}_t} |Du|^2 - \left(Du \cdot \vec{\theta}\right)^2 + \frac{u^2}{t^2} + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}} \end{aligned}$$

Notice, on the other hand, that the derivative of $t^{-2}J(\mathcal{C}_t)$ with respect to t throws out some similar terms to the ones we already have above,

$$\frac{t^3}{2}(t^{-2}J(\mathcal{C}_t))' = -J(\mathcal{C}_t) + \frac{t}{2} \int_{\partial\mathcal{C}_t} |Du|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}}.$$

Then by using that u is a minimizer in $\mathcal{C}_t$ we obtain,

$$\begin{aligned} 0 \leq J(u_t) - J(u) &= -J(\mathcal{C}_t) + \frac{t}{2} \int_{\partial\mathcal{C}_t} |Du|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}}, \\ &- \frac{t}{2} \int_{\partial\mathcal{C}_t} \left(Du \cdot \vec{\theta}\right)^2 - \frac{u^2}{t^2}, \\ &= \frac{t^3}{2}(t^{-2}J(\mathcal{C}_t))' - \frac{t}{2} \int_{\partial\mathcal{C}_t} \left(Du \cdot \vec{\theta}\right)^2 - \frac{u^2}{t^2} \end{aligned}$$

By writing,

$$\frac{u(t\vec{\theta})}{t} = \lim_{\varepsilon\to 0^+} \frac{1}{t} \int_{\varepsilon}^{t} Du\left(s\vec{\theta}\right) \cdot \vec{\theta} ds \text{ for } t\vec{\theta} \in \partial\mathcal{C}_t,$$

we obtain by Hölder's inequality that

$$
\begin{aligned}
0 &\le \lim_{\varepsilon\to 0^+} \frac{1}{t}\int_{\partial \mathcal{C}_1} \frac{1}{t}\int_{\varepsilon}^{t} \left(Du(s\vec{\theta})\cdot\vec{\theta}\right)^2 ds - \left(\frac{1}{t}\int_{\varepsilon}^{t} Du\left(s\vec{\theta}\right)\cdot\vec{\theta}\right)^2 , \\
&\le (t^{-2}J(\mathcal{C}_t))' + \lim_{\varepsilon\to 0^+}\int_{\partial\mathcal{C}_1}\left\{\frac{1}{t^2}\int_{\varepsilon}^{t}\left(Du(s\vec{\theta})\cdot\vec{\theta}\right)^2 ds - \frac{1}{t}\left(Du(t\vec{\theta})\cdot\vec{\theta}\right)^2\right\}, \\
&= \left(t^{-2}J(\mathcal{C}_t) - \lim_{\varepsilon\to 0^+}\frac{1}{t}\int_{\varepsilon}^{t}\int_{\partial\mathcal{C}_1}\left(Du(s\vec{\theta})\cdot\vec{\theta}\right)^2 ds\right)'
\end{aligned}
$$

This implies the monotonicity.

In case of having $W(\mathcal{C}_{r_1}) = W(\mathcal{C}_{r_2})$ the motononicity forces the equality also in the whole interval $[r_1, r_2]$. The use of Hölder's implies that for almost every $s \in [r_1, r_2]$, $Du(s\vec{\theta})\cdot\vec{\theta}$ is independent of s which is equivalent to the radial derivative of u being 0-homogeneous and u being 1-homogeneous. □

Proof of the second part in Corollary 9.3.6. All we have to check is that for any $r_1 < r_2$, $W(\mathcal{C}_{r_1}, u_0) \ge W(\mathcal{C}_{r_2}, u_0)$. From the rescaling property of the functional, Remark 12.4.6, we obtain that for $i = 1, 2$ we have that $W(\mathcal{C}_{\rho_k r_i}, u) = W(\mathcal{C}_{r_i}, u_{\rho_k}) \to W(\mathcal{C}_{r_i}, u_0)$. For each k there exists some $m_k \ge k$ such that $\rho_{m_k} r_2 < \rho_k r_1$ which implies that $W(\mathcal{C}_{\rho_{m_k} r_2}, u) \le W(\mathcal{C}_{\rho_k r_1}, u)$. By taking $k \to \infty$ we conclude the desired inequality. □

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